Unbounded perturbations of the generator domain

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Boundary perturbations of the shift semigroup

Let $(U, \|\cdot\|)$ be a Banach space. Given $z : [-1, +\infty) \to U$, $t \geq 0$ and $\mu : [-1, 0] \to \mathcal{L}(U)$ is of bounded variation with $\mu(0) = 0$.

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History function:

$$z_t : [-1, 0] \to U; \quad z_t(\theta) = z(t + \theta),\ \theta \in [-1, 0].$$

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History function:

$$z_t : [-1, 0] \to U; \quad z_t(\theta) = z(t + \theta),\ \theta \in [-1, 0].$$

Difference equation:

$$\begin{aligned} z(t) &= \int_{-1}^{0} d\mu(\theta) z(t + \theta), \quad t \geq 0, \\ z(\theta) &= \varphi(\theta) \text{ a.e. } \theta \in [-1, 0]. \end{aligned}$$

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$$v(t,\theta) := z(t+\theta), \quad \theta \in [-1,0],\ t \geq 0$$

solves

$$\begin{cases} \frac{\partial}{\partial t} v(t,\theta) &= \frac{\partial}{\partial \theta} v(t,\theta), \quad (t,\theta) \in [0,+\infty) \times [-1,0], \\ v(t,0) &= \int_{-1}^{0} d\mu(\theta) v(t,\theta), \quad t \geq 0, \\ v(0,\theta) &= \varphi(\theta), \qquad \theta \in [-1,0]. \end{cases} \tag{1}$$

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(1) is reformulated in the abstract from

$$\begin{cases} \dot{v}(t,\cdot) & = Lv(t,\cdot), \quad t \geq 0, \\ Gv(t,\cdot) & = Mv(t,\cdot), \quad t \geq 0, \\ v(0,\cdot) & = \varphi. \end{cases}$$

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$$\begin{cases} \frac{\partial z}{\partial t}(x,t) &= \frac{\partial^2 z}{\partial x^2}(x,t), \quad 0 < x < \pi, \quad t \geq 0, \\ \frac{\partial z}{\partial x}(0,t) &= \int_0^\pi \int_{-1}^0 d\mu(\theta) z(x,t+\theta)\, dx, \quad z(\pi,t) = 0,\ t \geq 0, \\ z(x,\theta) &= \varphi(x,\theta), \quad 0 < x < \pi,\ \theta \in [-1,0], \\ z(x,0) &= z^0(x), \quad 0 < x < \pi, \end{cases} \tag{2}$$

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where $\mu : [-1,0] \to \mathbb{R}$ is of bounded variation with $\mu(0) = 0$.

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Set
$v(x,t,\theta) = z(x,t+\theta) = z_t(x,\theta),\ t \geq 0,\ \theta \in [-1,0],\ x \in [0,\pi].$

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Set
$v(x,t,\theta) = z(x,t+\theta) = z_t(x,\theta),\ t \geq 0,\ \theta \in [-1,0],\ x \in [0,\pi]$.
Then,

$$\begin{cases} \frac{\partial v}{\partial t}(x,t,\theta) = \frac{\partial v}{\partial \theta}(x,t,\theta), & (t,\theta) \in [0,+\infty) \times [-1,0], \\ v(x,t,0) = z(x,t), & t \geq 0, \\ v(x,0,\theta) = \varphi(x,\theta), & \theta \in [-1,0]. \end{cases} \tag{3}$$

Boundary perturbations of the heat semigroup

Introducing the new state

$$w(t,x) = \begin{pmatrix} z(x,t) \\ z_t(x,\cdot) \end{pmatrix}, \qquad 0 < x < \pi, \quad t \geq 0,$$

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Introducing the new state

$$w(t,x) = \begin{pmatrix} z(x,t) \\ z_t(x,\cdot) \end{pmatrix}, \qquad 0 < x < \pi, \quad t \geq 0,$$

(2) can be reformulated as

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial}{\partial \theta} \end{pmatrix} w(x,t), \quad 0 < x < \pi,\ t \geq 0, \\ \begin{pmatrix} \frac{\partial z}{\partial x}(0,t) \\ z_t(x,0) \end{pmatrix} = \begin{pmatrix} \int_0^\pi \int_{-1}^0 d\mu(\theta) z_t(x,\theta)\, dx \\ z(x,t) \end{pmatrix}, \quad 0 < x < \pi, \quad t \geq 0, \\ z(\pi,t) = 0, \quad t \geq 0, \\ w(x,0) = \begin{pmatrix} z^0(x) \\ \varphi(x,\cdot) \end{pmatrix}. \end{cases} \tag{4}$$

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and

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\begin{aligned}
L_0\psi &= \frac{d^2\phi}{dx^2},\ G_0 f = \frac{d\phi}{dx}(0), \quad \phi \in Z_0 \\
M_0\psi &= \int_0^\pi \int_{-1}^0 d\mu(\theta)\psi(x,\theta)\,dx, \quad \psi \in W^{1,2}([-1,0],X_0).
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On the Hilbert spaces

$$X = X_0 \times L^2([-1,0],X_0),\ Z = Z_0 \times W^{1,2}([-1,0],X_0),\ U = \mathbb{C} \times \mathbb{C},$$

one defines the operators

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\begin{aligned}
L &= \begin{pmatrix} L_0 & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} : Z \to X, \\
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\begin{cases}
\dot{w}(t) = Lw(t), & t \geq 0, \\
Gw(t) = Mw(t), & t \geq 0, \\
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\end{cases}
\tag{5}
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where $L : Z \to X, \quad G, M : Z \to U$ are closed linear operators with $Z \subset X$, s.t.

- $L_0 = L,\ D(L_0) := \{z \in Z : Gz = 0\}$ generates a $C_0 - \text{semigroup}$ on X
- $G : Z \to U$ is surjective.

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or as an input–output boundary system

$$(IOBS)\begin{cases}\dot{z}(t) = Lz(t), & t \geq 0,\\ Gz(t) = u(t), & t \geq 0,\\ z(0) = z_0, & \\ y(t) = Mz(t), & t \geq 0\end{cases}$$

with the feedback law $u(t) = y(t)$.

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the well-posedness of (CP) (and equivalently $(IOBS)$) was proved using two approaches:

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Lemma 1:

For $\lambda \in \rho(L_0)$,

(i) $Z = D(L_0) \oplus \ker(\lambda - L)$;

(ii) $G|_{\ker(\lambda - L)}$ is invertible and

$$\mathbb{D}_\lambda := \left(G|_{\ker(\lambda - L)}\right)^{-1} : U \to \ker(\lambda - L) \subseteq Z \text{ is bounded.}$$

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Lemma 1 implies that

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So, (BS) can be reformulated as

$$\dot{z}(t) = L_0 z(t) + Bu(t), \quad t \geq 0, \quad z(0) = z_0.$$

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Hence, $(IOBS)$ can be reformulated as

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Yosida extensions

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$$\begin{aligned} C_\Lambda x &:= \lim_{\lambda\to+\infty} C\lambda(\lambda - L_0)^{-1}x,\ x \in D(C_\Lambda), \\ D(C_\Lambda) &:= \{x \in X : \lim_{\lambda\to+\infty} C\lambda(\lambda - L_0)^{-1}x \text{ exists in } U\}. \end{aligned}$$

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Lemma 2:
Assume, for some $\lambda \in \rho(L_0)$,

$$\mathrm{Range}((\lambda - (L_0)_{-1})^{-1}B) \subset D(C_\Lambda)$$

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Then $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X satisfying $T(s)z_0 \in D(C_\Lambda)$, $\forall z_0 \in X$ and a.e. $s \geq 0$.

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In addition

$$T(t)z_0=T_0(t)z_0+\int_0^t (T_0)_{-1}(t-s)BC_\Lambda T(s)z_0\,ds,\ \forall z_0\in X,\ t\geq 0.$$

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On the other hand, for any $\lambda \in \rho(L_0)$ we have

$$\lambda \in \rho(A) \iff 1 \in \rho(\mathbb{D}_\lambda M) \iff 1 \in \rho(M\mathbb{D}_\lambda).$$

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Then $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X satisfying $T(s)z_0 \in D(C_\Lambda)$, $\forall z_0 \in X$ and a.e. $s \geq 0$.
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On the other hand, for any $\lambda \in \rho(L_0)$ we have

$$\lambda \in \rho(A) \iff 1 \in \rho(\mathbb{D}_\lambda M) \iff 1 \in \rho(M\mathbb{D}_\lambda).$$

Finally, for $\lambda \in \rho(L_0) \cap \rho(A)$,

$$(\lambda - A)^{-1} = (I - \mathbb{D}_\lambda M)^{-1}(\lambda - L_0)^{-1}.$$

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A Theorem due to G. Weiss (MCSS 1994), and O.J. Staffans 2005 (for general Banach spaces) imply

$$\begin{aligned} A^I x &:= ((L_0)_{-1} + BC_\Lambda)x, \\ x \in D(A^I) &:= \left\{x \in D(C_\Lambda) : ((L_0)_{-1} + BC_\Lambda)x \in X\right\} \end{aligned}$$

generates a C_0-semigroup on X.

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generates a C_0-semigroup on X.
Now, it suffices to show that $A = A^I$.

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