Hamiltonian Dynamics, Hierarchies and Liouville's equation:

A statistical point of view

Zied Ammari
joint works with F. Nier, Q. Liard and C. Rouffort

Rennes University, IRMAR

12th of December 2016, Marrakech

Hamiltonian systems

- Equation of motion: a Hamiltonian system with an infinite number of degrees of freedom is described by pairs of momentum-coordinate canonical variables $(p_1, q_1 \cdots, p_n, q_n, \cdots)$. The equation of motion is derived from a classical Hamiltonian:

$$\mathcal{H}(p, q) = \mathcal{H}(p_1, q_1, \cdots, p_n, q_n, \cdots)$$

$$\dot{q}_j = \frac{\delta \mathcal{H}}{\delta p_j}, \qquad \dot{p}_j = -\frac{\delta \mathcal{H}}{\delta q_j}, \qquad j = 1, \cdots \tag{1}$$

Hamiltonian systems

- Equation of motion: a Hamiltonian system with an infinite number of degrees of freedom is described by pairs of momentum-coordinate canonical variables $(p_1, q_1 \cdots, p_n, q_n, \cdots)$. The equation of motion is derived from a classical Hamiltonian:

$$\mathcal{H}(p,q) = \mathcal{H}(p_1, q_1, \cdots, p_n, q_n, \cdots)$$

$$\dot{q}_j = \frac{\delta \mathcal{H}}{\delta p_j}, \qquad \dot{p}_j = -\frac{\delta \mathcal{H}}{\delta q_j}, \qquad j = 1, \cdots \tag{1}$$

- Descriptions: The dynamics of a Hamiltonian system can be described either by:
  1- The evolution of dynamical states: phase-space points.
  2- The evolution of statistical states: probability distributions.

Hamiltonian systems

- Equation of motion: a Hamiltonian system with an infinite number of degrees of freedom is described by pairs of momentum-coordinate canonical variables $(p_1, q_1 \cdots, p_n, q_n, \cdots)$. The equation of motion is derived from a classical Hamiltonian:

$$\mathcal{H}(p, q) = \mathcal{H}(p_1, q_1, \cdots, p_n, q_n, \cdots)$$

$$\dot{q}_j = \frac{\delta \mathcal{H}}{\delta p_j}, \qquad \dot{p}_j = -\frac{\delta \mathcal{H}}{\delta q_j}, \qquad j = 1, \cdots \tag{1}$$

- Descriptions: The dynamics of a Hamiltonian system can be described either by:
  1- The evolution of dynamical states: phase-space points.
  2- The evolution of statistical states: probability distributions.
- Approaches:
  1- Quantitative study: Well-posedness, stability, scattering $\cdots$
  2- Qualitative study: Ergodicity, chaos, asymptotic behavior $\cdots$

Liouville equation

- In finite dimension: The time evolution of a probability distribution function $\varrho(p, q, t)$ describing a Hamiltonian system at time t is governed by the Liouville equation:

$$\frac{\partial \varrho}{\partial t} + \{\varrho, \mathcal{H}\} = 0 \,, \tag{2}$$

with the Poisson bracket defined as:

$$\{\varrho, \mathcal{H}\} = \sum_{i=1}^{n} \left[\frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \varrho}{\partial q_i} - \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial \varrho}{\partial p_i} \right] .$$

Liouville equation

- In finite dimension: The time evolution of a probability distribution function $\varrho(p, q, t)$ describing a Hamiltonian system at time t is governed by the Liouville equation:

$$\frac{\partial \varrho}{\partial t} + \{\varrho, \mathcal{H}\} = 0\,, \tag{2}$$

with the Poisson bracket defined as:

$$\{\varrho, \mathcal{H}\} = \sum_{i=1}^{n} \left[\frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \varrho}{\partial q_i} - \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial \varrho}{\partial p_i} \right] .$$

- The Liouville's theorem: says that the distribution function is constant along any trajectory in the phase space, i.e.,

$$\frac{d}{dt} \varrho(p_t, q_t, t) = 0\,.$$

Liouville equation

- In finite dimension: The time evolution of a probability distribution function $\varrho(p, q, t)$ describing a Hamiltonian system at time t is governed by the Liouville equation:

$$\frac{\partial \varrho}{\partial t} + \{\varrho, \mathcal{H}\} = 0\,, \tag{2}$$

with the Poisson bracket defined as:

$$\{\varrho, \mathcal{H}\} = \sum_{i=1}^{n} \left[\frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \varrho}{\partial q_i} - \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial \varrho}{\partial p_i} \right] .$$

- The Liouville's theorem: says that the distribution function is constant along any trajectory in the phase space, i.e.,

$$\frac{d}{dt} \varrho(p_t, q_t, t) = 0\,.$$

- The method of characteristics: if the Hamiltonian is sufficiently smooth, then the density function $\varrho(p, q, t)$ is uniquely determined by its initial value $\varrho(p, q, 0)$, i.e.,

$$\varrho(p, q, t) = \varrho(\Phi_t^{-1}(p, q), 0)\,.$$

Our goal:

- The Liouville's equation is the natural ground for statistical mechanics.
- The method of characteristics relates in finite dimension the individual solutions of the classical Hamiltonian system (1) (ODE) and the statistical (probability measure) solutions of the Liouville equation.

- Give a rigorous meaning of a Liouville equation for Hamiltonian PDEs or more generally dynamical systems with infinite degrees of freedom.

Our goal:

- The Liouville's equation is the natural ground for statistical mechanics.
- The method of characteristics relates in finite dimension the individual solutions of the classical Hamiltonian system (1) (ODE) and the statistical (probability measure) solutions of the Liouville equation.

- Give a rigorous meaning of a Liouville equation for Hamiltonian PDEs or more generally dynamical systems with infinite degrees of freedom.
- Establish uniqueness results for the Liouville equation under wide general assumptions.

Our goal:

- The Liouville's equation is the natural ground for statistical mechanics.
- The method of characteristics relates in finite dimension the individual solutions of the classical Hamiltonian system (1) (ODE) and the statistical (probability measure) solutions of the Liouville equation.

- Give a rigorous meaning of a Liouville equation for Hamiltonian PDEs or more generally dynamical systems with infinite degrees of freedom.
- Establish uniqueness results for the Liouville equation under wide general assumptions.
- Provide an alternative framework for a statistical theory of PDEs.

Our goal:

- The Liouville's equation is the natural ground for statistical mechanics.
- The method of characteristics relates in finite dimension the individual solutions of the classical Hamiltonian system (1) (ODE) and the statistical (probability measure) solutions of the Liouville equation.

- Give a rigorous meaning of a Liouville equation for Hamiltonian PDEs or more generally dynamical systems with infinite degrees of freedom.
- Establish uniqueness results for the Liouville equation under wide general assumptions.
- Provide an alternative framework for a statistical theory of PDEs.
- Application to the study of QFT. In particular, the following examples:
  - Many-Body theory (N-body Schrödinger operators)
  - Relativistic Quantum field theory ($(\varphi)^4_2, P(\varphi)_2$ models)
  - Non-relativistic Quantum field theory (Nelson model)
  - Quantum electrodynamics (Pauli-Fierz models)

Initial value problem

Consider a rigged Hilbert space $\mathscr{Z}_1 \subset \mathscr{Z}_0 \subset \mathscr{Z}_1'$ such that $(\mathscr{Z}_1, \mathscr{Z}_0)$ is a pair of complex *separable* Hilbert spaces, $\mathscr{Z}_1$ is densely continuously embedded in $\mathscr{Z}_0$ and $\mathscr{Z}_1'$ is the dual of $\mathscr{Z}_1$ with respect to the duality bracket extending the inner product $\langle \cdot, \cdot \rangle_{\mathscr{Z}_0}$.

- Let $v : \mathbb{R} \times \mathscr{Z}_1 \to \mathscr{Z}_1'$ be a non-autonomous *continuous* vector field, i.e. $v \in C(\mathbb{R} \times \mathscr{Z}_1, \mathscr{Z}_1')$, such that v is bounded on bounded sets of $\mathbb{R} \times \mathscr{Z}_1$. We shall consider the following initial value (or Cauchy) problem on an open interval $I \subset \mathbb{R}$:

$$\begin{cases} \dot{\gamma}(t) = v(t, \gamma(t)) \,, \\ \gamma(s) = x \in \mathscr{Z}_1 \,, \qquad s \in I \,, \end{cases} \tag{3}$$

Initial value problem

Consider a rigged Hilbert space $\mathscr{Z}_1 \subset \mathscr{Z}_0 \subset \mathscr{Z}_1'$ such that $(\mathscr{Z}_1, \mathscr{Z}_0)$ is a pair of complex *separable* Hilbert spaces, $\mathscr{Z}_1$ is densely continuously embedded in $\mathscr{Z}_0$ and $\mathscr{Z}_1'$ is the dual of $\mathscr{Z}_1$ with respect to the duality bracket extending the inner product $\langle \cdot, \cdot \rangle_{\mathscr{Z}_0}$.

- Let $v : \mathbb{R} \times \mathscr{Z}_1 \to \mathscr{Z}_1'$ be a non-autonomous *continuous* vector field, i.e. $v \in C(\mathbb{R} \times \mathscr{Z}_1, \mathscr{Z}_1')$, such that v is bounded on bounded sets of $\mathbb{R} \times \mathscr{Z}_1$. We shall consider the following initial value (or Cauchy) problem on an open interval $I \subset \mathbb{R}$:

$$\begin{cases} \dot{\gamma}(t) = v(t, \gamma(t))\,, \\ \\ \gamma(s) = x \in \mathscr{Z}_1\,, \qquad s \in I\,, \end{cases} \tag{3}$$

- (i) A weak solution of the above initial value problem on I is a function $I \ni t \to \gamma(t)$ belonging to the space $L^\infty(I, \mathscr{Z}_1) \cap W^{1,\infty}(I, \mathscr{Z}_1')$ satisfying (3) for a.e. $t \in I$ and for some $s \in I$.

Initial value problem

Consider a rigged Hilbert space $\mathscr{Z}_1 \subset \mathscr{Z}_0 \subset \mathscr{Z}_1'$ such that $(\mathscr{Z}_1, \mathscr{Z}_0)$ is a pair of complex *separable* Hilbert spaces, $\mathscr{Z}_1$ is densely continuously embedded in $\mathscr{Z}_0$ and $\mathscr{Z}_1'$ is the dual of $\mathscr{Z}_1$ with respect to the duality bracket extending the inner product $\langle \cdot, \cdot \rangle_{\mathscr{Z}_0}$.

- Let $v : \mathbb{R} \times \mathscr{Z}_1 \to \mathscr{Z}_1'$ be a non-autonomous *continuous* vector field, i.e. $v \in C(\mathbb{R} \times \mathscr{Z}_1, \mathscr{Z}_1')$, such that v is bounded on bounded sets of $\mathbb{R} \times \mathscr{Z}_1$. We shall consider the following initial value (or Cauchy) problem on an open interval $I \subset \mathbb{R}$:

$$\begin{cases} \dot\gamma(t) = v(t, \gamma(t))\,, \\ \\ \gamma(s) = x \in \mathscr{Z}_1\,, \qquad s \in I\,, \end{cases} \tag{3}$$

- (i) A weak solution of the above initial value problem on I is a function $I \ni t \to \gamma(t)$ belonging to the space $L^\infty(I, \mathscr{Z}_1) \cap W^{1,\infty}(I, \mathscr{Z}_1')$ satisfying (3) for a.e. $t \in I$ and for some $s \in I$.
- (ii) A strong solution of the above initial value problem on I is a function $I \ni t \to \gamma(t)$ belonging to the space $\gamma \in C(I, \mathscr{Z}_1) \cap C^1(I, \mathscr{Z}_1')$ satisfying (3) for all $t \in I$ and for some $s \in I$.

Well-posedness

We say that the initial value problem (3) is locally well posed (LWP) in $\mathscr{Z}_1$ if:

(i) Weak uniqueness: Any two weak solutions of (3), defined on the same open interval I and satisfying the same initial condition, coincide.

(ii) Strong existence: For any $x \in \mathscr{Z}_1$ and $s \in \mathbb{R}$, there exists a non-empty open interval I containing s such that a strong solution of (3) defined on I exists.

(iii) Blowup alternative: Let $(T_{min}(x,s), T_{max}(x,s))$ be the maximal interval of existence of a strong solution of (3). If $T_f = T_{max}(x,s) < +\infty$ (resp. $T_i = T_{min}(x,s) > -\infty$) then,

$$\lim_{t \uparrow T_f} ||\gamma(t)||_{\mathscr{Z}_1} = +\infty\,, \quad (\text{resp. } \lim_{t \downarrow T_i} ||\gamma(t)||_{\mathscr{Z}_1} = +\infty)\,.$$

(iv) Continuous dependence on initial data: If $x_n \to x$ in $\mathscr{Z}_1$ and $J \subset (T_{min}(x,s), T_{max}(x,s))$ is a closed interval, then for n large enough the strong solutions γ_n of (3) provided by (ii) with $\gamma_n(s) = x_n$ are defined on J and satisfy $\gamma_n \underset{n\to\infty}{\to} \gamma$ in $C(J, \mathscr{Z}_1)$.

If $I = \mathbb{R}$ in (ii) for any $x \in \mathscr{Z}_1$ and any $s \in \mathbb{R}$, we say that the initial value problem is globally well-posed (GWP).

Projective setting

Let $\mathscr{Z}$ be a complex separable Hilbert space endowed with its euclidian structure $\mathrm{Re}\langle\cdot,\cdot\rangle_{\mathscr{Z}}$, denoted for shortness by $\langle\cdot,\cdot\rangle_{\mathscr{Z},\mathbb{R}}$. Consider $\mathscr{Z}_{\mathbb{R}} := \mathscr{Z}$ as a real Hilbert space and let $\Pi_n(\mathscr{Z}_{\mathbb{R}})$ be the set of all projections $\pi : \mathscr{Z}_{\mathbb{R}} \to \mathbb{R}^n$ defined by

$$\pi(x) = (\langle x, e_1\rangle_{\mathscr{Z},\mathbb{R}}, \cdots, \langle x, e_n\rangle_{\mathscr{Z},\mathbb{R}})\,, \tag{4}$$

where $\{e_1, \cdots, e_n\}$ is any orthonormal family of $\mathscr{Z}_{\mathbb{R}}$. We denote by $\mathscr{C}^\infty_{0,cyl}(\mathscr{Z})$ the space of functions $\varphi = \psi \circ \pi$ with $\pi \in \Pi_n(\mathscr{Z}_{\mathbb{R}})$ for some $n \in \mathbb{N}$ and $\psi \in \mathscr{C}^\infty_0(\mathbb{R}^n)$. In particular, one can check that the gradient (or the $\mathbb{R}$-differential) of φ is equal to

$$\nabla\varphi = \pi^T \circ \nabla\psi \circ \pi,$$

where π^T denotes the transpose map of π. We equally define, for any open interval $I \subset \mathbb{R}$, the space $\mathscr{C}^\infty_{0,cyl}(I \times \mathscr{Z})$ as the set of functions $\varphi(t,x) = \psi(t, \pi(x))$ with $\psi \in \mathscr{C}^\infty_0(\mathbb{R}^{n+1})$ and $\pi \in \Pi_n(\mathscr{Z}_{\mathbb{R}})$.

Liouville equation in infinite dim

Let $v : \mathbb{R} \times \mathscr{Z}_1 \to \mathscr{Z}_1'$ a continuous vector field. We consider the following Liouville's equation defined in a bounded open interval $I \subset \mathbb{R}$,

$$\partial_t \mu_t + \nabla^T (v.\mu_t) = 0\,,$$

understood, in a weak sense, as the integral equation:
$\forall \varphi \in \mathscr{C}^\infty_{0,cyl}(I \times \mathscr{Z}_1')$,

$$\int_I \int_{\mathscr{Z}_1'} \partial_t \varphi(t,x) + \mathrm{Re}\langle v(t,x), \nabla\varphi(t,x)\rangle_{\mathscr{Z}_1'} \, d\mu_t(x)\, dt = 0\,. \tag{5}$$

In order that the above problem makes sense we assume that $\mu_t \in \mathfrak{P}(\mathscr{Z}_1)$ for all $t \in I$. So the integration with respect to μ_t is taken on the set $\mathscr{Z}_1$ where the integrand is well defined. We also assume two more conditions on $t \to \mu_t$, namely we require that

$$\int_I \int_{\mathscr{Z}_1} \|v(t,x)\|_{\mathscr{Z}_1'} \, d\mu_t(x)\, dt < \infty\,, \tag{6}$$

the curve $I \ni t \to \mu_t$ is weakly narrowly continuous in $\mathfrak{P}(\mathscr{Z}_1')$

Main Results:

Theorem (Am.-Liard)

Let $v : \mathbb{R} \times \mathscr{Z}_1 \to \mathscr{Z}_1'$ *be a (non-autonomous) continuous vector field such that* v *is bounded on bounded sets. Let* $t \in I \to \mu_t \in \mathfrak{P}(\mathscr{Z}_1)$ *be a weakly narrowly continuous solution in* $\mathfrak{P}(\mathscr{Z}_1')$ *of the Liouville equation* (5) *defined on an open bounded interval* I*. Assume additionally that:*

(i) *There exists a ball* B *of* $\mathscr{Z}_1$ *such that* $\mu_t(B) = 1$ *for all* $t \in I$*.*

(ii) *The initial value problem* (3) *is (LWP) in* $\mathscr{Z}_1$*.*

Then for any $s \in I$*, the maximal existence interval*

$$(T_{min}(x,s), T_{max}(x,s)) \supseteq I \qquad \text{for } \mu_s - \text{a.e. } x \in \mathscr{Z}_1.$$

Moreover,

$$\mu_t = \Phi(t,s)_\sharp \mu_s\,, \qquad \text{for all } t \in I$$

with $\Phi(t,s)$ *is the local flow of the initial value problem* (3)*. Additionally, if the curve* $t \to \mu_t$ *is defined on* $\mathbb{R}$ *and the above assumptions still satisfied for any arbitrary bounded open interval* $I \subset \mathbb{R}$*, then* $\mu_t = \Phi(t,s)_\sharp \mu_s$ *for all* $t, s \in \mathbb{R}$*.*

Main Results:

Theorem (Am.-Liard)

Let $v : \mathbb{R} \times \mathscr{Z}_1 \to \mathscr{Z}_0$ be a (non-autonomous) continuous vector field such that for any $M > 0$ and any bounded interval $J \subset \mathbb{R}$, there exists $C(M, J) > 0$ satisfying:

$$||v(t,x) - v(t,y)||_{\mathscr{Z}_0} \leq C(M,J)\,(||x||^2_{\mathscr{Z}_1} + ||y||^2_{\mathscr{Z}_1})\,||x-y||_{\mathscr{Z}_0}\,, \tag{7}$$

for all $t \in J$ and $x, y \in \mathscr{Z}_1$ such that $||x||_{\mathscr{Z}_0}, ||y||_{\mathscr{Z}_0} \leq M$. Let $t \in I \to \mu_t \in \mathfrak{P}(\mathscr{Z}_1)$ be a weakly narrowly continuous solution in $\mathfrak{P}(\mathscr{Z}_1')$ of the Liouville equation (5) defined on an open bounded interval I. Assume additionally that:

- (i) *There exists $C > 0$ such that $\int_I \int_{\mathscr{Z}_1} ||x||^2_{\mathscr{Z}_1} d\mu_t(x)dt \leq C$.*
- (ii) *There exists an open Ball B of $\mathscr{Z}_0$ such that $\mu_t(B) = 1$ for all $t \in I$.*
- (iii) *For $s \in I$ and any $x \in \mathscr{Z}_1 \cap B$ there exists a strong solution of (3) defined on $\overline{I}$ with (LWP-iv) satisfied.*

Then $\mu_t = \Phi(t,s)_\sharp \mu_s$ for all $t \in I$ with $\Phi(t,s)$ is the local flow of the initial value problem (3).

Application to Hamiltonian PDEs

Consider a Hamiltonian PDE with a real-valued energy functional,

$$h(z,\bar{z}) = \langle z, Az\rangle_{\mathscr{Z}_0} + h_I(z,\bar{z})\,, \tag{8}$$

where $\mathscr{Z}_0$ is a complex separable Hilbert space, A is a non-negative self-adjoint operator. Take $\mathscr{Z}_1 = D(A^{1/2})$ the energy space and $\mathscr{Z}_1'$ its dual. Assume that the energy (8) is well-defined on $\mathscr{Z}_1$ and that h admits directional derivatives,

$$\partial_{\bar{z}} h(x,\bar{x})[u] := \frac{d}{d\bar{\lambda}} h(x+\lambda u, \overline{x+\lambda u})_{|\lambda=0}\,,$$

such that the map $x \in \mathscr{Z}_1 \to \partial_{\bar{z}} h(x,\bar{x}) \in \mathscr{Z}_1'$ is continuous and bounded on bounded sets. The Hamiltonian equation reads,

$$i\partial_t u = \partial_{\bar{z}} h(u,\bar{u})\,. \tag{9}$$

Thus, Theorem 1 can be applied to the Hamiltonian equation (9) if either (LWP) or (GWP) holds true in the energy space $\mathscr{Z}_1$.

Nonlinear Schrödinger equation:

Consider the NLS equation in dimension d with energy functional,

$$h(z,\bar{z}) = \langle z, -\Delta_x z\rangle_{L^2(\mathbb{R}^d)} + \frac{2\lambda}{2+\alpha}\int_{\mathbb{R}^d} |z(x)|^{\alpha+2}\,dx\,, \tag{10}$$

such that $\lambda \in \mathbb{C}$ and

$$2 \leq \alpha < \frac{4}{d-2} \quad (2 \leq \alpha < \infty \text{ if } d = 1,2)\,.$$

The related initial value problem is (LWP) in $H^1(\mathbb{R}^d)$. Hence, Theorem 1 applies to this case. The derivation of such equation from quantum many-body dynamics, for $\alpha = 2$, is proved for instance in Erdös-Schlein-Yau and [ABGT04, AB12].

Non-relativistic Hartree equation

The energy functional of the Hartree equation is

$$h(z,\bar{z}) = \langle z, -\Delta_x + V(x)\, z\rangle_{L^2(\mathbb{R}^d)} + \iint_{\mathbb{R}^d\times\mathbb{R}^d} |z(x)|\, |z(y)|^2\, W(x-y)\, dxdy\,, \tag{11}$$

where $W : \mathbb{R}^d \to \mathbb{R}$ is an even measurable function and V is a real-valued potential both satisfying the following assumptions,

$$V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ p \geq 1,\ \ p > \frac{d}{2},$$

$$W \in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),\ q \geq 1,\ \ q \geq \frac{d}{2}\ (\text{and } q > 1 \text{ if } d = 2)\,.$$

The vector field $v(t,z) := W * |z|^2 z : D(A^{1/2}) \to L^2(\mathbb{R}^d)$ verifies the estimate (7). The global well-posedness on $D(A^{1/2})$, conservation of energy and charge of the Hartree equation

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{t=0} = z_0, \end{cases}$$

are proved in [Caz03]. Therefore, Theorem 2 applies here and we can take $W = \frac{\lambda}{|x|^\alpha}$ with $\alpha < 2$, $\lambda \in \mathbb{R}$ and $d = 3$.

Klein-Gordon equation

Consider the classical Klein-Gordon energy functional

$$\mathscr{H}(\varphi,\pi) = \frac{1}{2}\langle \varphi, -\Delta + m^2\ \varphi\rangle^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}||\pi(x)||^2_{L^2(\mathbb{R}^3)} + \frac{1}{4}\int_{\mathbb{R}^3} \varphi^4(x)dx \tag{12}$$

where φ, π are the real fields and $m > 0$. Writing the above system with the complex fields $(z(\cdot), \bar{z}(\cdot))$:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\bar{z}(k)e^{-ikx} + z(k)e^{-ikx}) \frac{dk}{\sqrt{2\omega(k)}}\,,$$

$$\pi(x) = \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\bar{z}(k)e^{-ikx} - z(k)e^{-ikx}) \sqrt{\frac{\omega(k)}{2}}\, dk\,,$$

we obtain the equivalent PDE,

$$i\partial_t z = \omega(k)z + \frac{1}{\sqrt{2\omega(k)}}\mathscr{F}(\varphi^3)\,, \tag{13}$$

We know that the Klein-Gordon equation (13) is (GWP) in $\mathscr{Z}_1 = D(\sqrt{\omega})$ according to Ginibre-Velo. Hence, Theorem 1 is applicable. The derivation of a Klein-Gordon equation with nonlocal nonlinearity from the $P(\varphi)_2$ quantum field theory is established in [AZ12, Hep74].

We denote by $\Gamma_I(E)$ the space of all continuous curves from $\bar{I}$ into $(E, ||\cdot||_E)$ endowed with the sup norm,

$$||\gamma||_{\Gamma_I(E)} = \sup_{t\in\bar{I}} ||\gamma(t)||_E \,.$$

and define the metric space

$$\mathfrak{X} = \left(\mathscr{Z}_1' \times \Gamma_I(\mathscr{Z}_1'), ||\cdot||_{\mathscr{Z}_{1,w}'} + ||\cdot||_{\Gamma_I(\mathscr{Z}_{1,w}')}\right) \tag{14}$$

Proposition

Let $v : \mathbb{R} \times \mathscr{Z}_1 \to \mathscr{Z}_1'$ be a (non-autonomous) Borel vector field such that v is bounded on bounded sets. Let $t \in I \to \mu_t \in \mathfrak{P}(\mathscr{Z}_1)$ be a weakly narrowly continuous solution in $\mathfrak{P}(\mathscr{Z}_1')$ of the Liouville equation (5) defined on an open bounded interval I with a vector field satisfying the scalar velocity estimate (6). Then there exists a Borel probability measure η, on the space $\mathfrak{X}$ given in (14), satisfying:

1. *η is concentrated on the set of $(x, \gamma) \in \mathscr{Z}_1 \times \Gamma_I(\mathscr{Z}_1')$ such that the curves $\gamma \in W^{1,1}(I, \mathscr{Z}_1')$ are solutions of the initial value problem $\dot{\gamma}(t) = v(t, \gamma(t))$ for a.e. $t \in I$ and $\gamma(t) \in \mathscr{Z}_1$ for a.e. $t \in I$ with $\gamma(s) = x \in \mathscr{Z}_1$ for some fixed $s \in I$.*
2. *$\mu_t = (e_t)_\sharp \eta$ for any $t \in I$.*

Further consequences:

There are several interesting consequences of our main results (Thm. 1 and 2).

- Uniqueness results for Liouville equation (Thm. 1) imply uniqueness results for the Gross-Pitaeivskii hierarchy. The main point is a conservation law with respect to the gauge group $U(1)$.

Further consequences:

There are several interesting consequences of our main results (Thm. 1 and 2).

- Uniqueness results for Liouville equation (Thm. 1) imply uniqueness results for the Gross-Pitaeivskii hierarchy. The main point is a conservation law with respect to the gauge group $U(1)$.
- Existence of stationary solutions of the Liouville equation and application of Poincaré recurrence lemma.

Further consequences:

There are several interesting consequences of our main results (Thm. 1 and 2).

- Uniqueness results for Liouville equation (Thm. 1) imply uniqueness results for the Gross-Pitaeivskii hierarchy. The main point is a conservation law with respect to the gauge group $U(1)$.
- Existence of stationary solutions of the Liouville equation and application of Poincaré recurrence lemma.
- Study of non-conservative dynamical systems: observability and controlability.

Further consequences:

There are several interesting consequences of our main results (Thm. 1 and 2).

- Uniqueness results for Liouville equation (Thm. 1) imply uniqueness results for the Gross-Pitaeivskii hierarchy. The main point is a conservation law with respect to the gauge group $U(1)$.
- Existence of stationary solutions of the Liouville equation and application of Poincaré recurrence lemma.
- Study of non-conservative dynamical systems: observability and controlability.
- Use Thm. 1 as a globalization tool for local solutions almost everywhere with respect to a stationary solution.

GP Hierarchy

The Gross-Pitaeivskii hierarchy satisfied by reduced density matrices $(\gamma^k)_{k\in\mathbb{N}} \in \mathcal{L}^1(L^2_s(\mathbb{R}^{2dk}))$, is:

$$\begin{cases} i\partial_t \gamma^k = \sum_{j=1}^{k} \big[-\Delta_{x_j}, \gamma^k \big] + B_{k+1}\gamma^{k+1} \\ \gamma^k_{t=0} = \gamma^k_0, \end{cases}$$

where

$$\begin{aligned} B_{k+1}\gamma^{(k+1)} &= B^+_{k+1}\gamma^{(k+1)} - B^-_{k+1}\gamma^{k+1} \\ &= \sum_{j=1}^{k} B^+_{j:k+1}\gamma^{k+1} - \sum_{j=1}^{k} B^-_{j:k+1}\gamma^{k+1}, \end{aligned}$$

given by their kernels for $0 \le j \le k$

1. $B^+_{j:k+1}\gamma^{(k+1)}(t, X_k, X'_k) = \gamma^{(k+1)}(t, X_k, x_j, X'_k, x_j)$;
2. $B^-_{j:k+1}\gamma^{(k+1)}(t, X_k, X'_k) = \gamma^{(k+1)}(t, X_k, x'_j, X'_k, x'_j)$.

GP Hierarchy

The Gross-Pitaeivskii hierarchy satisfied by reduced density matrices $(\gamma^k)_{k\in\mathbb{N}} \in \mathcal{L}^1(L^2_s(\mathbb{R}^{2dk}))$, is:

$$\begin{cases} i\partial_t \gamma^k = \sum_{j=1}^{k} \big[-\Delta_{x_j}, \gamma^k \big] + B_{k+1}\gamma^{k+1} \\ \gamma^k_{t=0} = \gamma^k_0, \end{cases}$$

where

$$\begin{aligned} B_{k+1}\gamma^{(k+1)} &= B^+_{k+1}\gamma^{(k+1)} - B^-_{k+1}\gamma^{k+1} \\ &= \sum_{j=1}^{k} B^+_{j:k+1}\gamma^{k+1} - \sum_{j=1}^{k} B^-_{j:k+1}\gamma^{k+1}, \end{aligned}$$

given by their kernels for $0 \leq j \leq k$

1. $B^+_{j:k+1}\gamma^{(k+1)}(t, X_k, X'_k) = \gamma^{(k+1)}(t, X_k, x_j, X'_k, x_j)$;
2. $B^-_{j:k+1}\gamma^{(k+1)}(t, X_k, X'_k) = \gamma^{(k+1)}(t, X_k, x'_j, X'_k, x'_j)$.

Theorem (Am.-Liard-Rouffort)

There is equivalence between the GP hierarchy and Liouville's equation

Z. Ammari and S. Breteaux.
Propagation of chaos for many-boson systems in one dimension with a point pair-interaction.
Asymptot. Anal., 76(3-4):123–170, 2012.

R. Adami, Claude Bardos, F. Golse, and A. Teta.
Towards a rigorous derivation of the cubic NLSE in dimension one.
Asymptot. Anal., 40(2):93–108, 2004.

Z. Ammari and M. Zerzeri.
On the classical limit of self-interacting quantum field Hamiltonians with cutoffs.
October 2012.

T. Cazenave.
Semilinear Schrödinger equations, volume 10 of *Courant Lecture Notes in Mathematics*.
New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

K. Hepp.
The classical limit for quantum mechanical correlation functions.
Comm. Math. Phys., 35:265–277, 1974.