# On resolvent conditions for the control of linear PDEs

Luc Miller

Université Paris Ouest Nanterre La Défense, France

*Marrakesh Workshop on Control*, december 13, 2016

L. M., *Resolvent conditions for the control of unitary groups and their approximations*, Journal of Spectral Theory 2 (2012).

Thomas Duyckaerts & L. M., *Resolvent conditions for the control of parabolic equations*, Journal of Functional Analysis 263 (2012).

M. Tucsnak & G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Advanced Texts: Basel Textbooks (2009).

Camille Laurent, *Internal control of the Schrödinger equation*, Mathematical Control and Related Fields 4 (2014).

# Outline

Control of the temperature $f$ in a smooth domain $M \subset \mathbb{R}^d$ (Dirichlet), from a chosen source $u$ acting in an open subset $\Omega \subset M$ during a time $T$.

My notation : $\Omega$ is also the operator $(\Omega v)(x) = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$

Norm : $\|v\|^2 = \int_M |v(x)|^2 dx$, so that $\|\Omega v\|^2 = \int_\Omega |v(x)|^2 dx$.

## Null-controllability in time $T$

The heat O.D.E. in $\mathcal{E} = L^2(M)$ with input $u \in L^2(\mathbb{R}; \mathcal{E})$: $\partial_t f - \Delta f = \Omega u$.

$$\forall f(0) \in \mathcal{E},\ \exists u,\ \text{such that} \quad f(T) = 0$$

Control of the temperature $f$ in a smooth domain $M \subset \mathbb{R}^d$ (Dirichlet), from a chosen source $u$ acting in an open subset $\Omega \subset M$ during a time $T$.

My notation : $\Omega$ is also the operator $(\Omega v)(x) = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$

Norm : $\|v\|^2 = \int_M |v(x)|^2 dx$, so that $\|\Omega v\|^2 = \int_\Omega |v(x)|^2 dx$.

## Null-controllability in time $T$ (at cost $\kappa_T$)

The heat O.D.E. in $\mathcal{E} = L^2(M)$ with input $u \in L^2(\mathbb{R}; \mathcal{E})$: $\partial_t f - \Delta f = \Omega u$.

$$\forall f(0) \in \mathcal{E}, \ \exists u, \text{ such that } \quad f(T) = 0 \text{ and } \int_0^T \|u(t)\|^2 dt \leqslant \kappa_T \|f(0)\|^2.$$

$$\Updownarrow$$

## Final-observability in time $T$ (at cost $\kappa_T$)

$$(FinalObs) \quad \|e^{T\Delta} v\|^2 \leqslant \kappa_T \int_0^T \|\Omega e^{t\Delta} v\|^2 dt, \quad v \in \mathcal{E}.$$

$t \mapsto f(t) = e^{t\Delta} v$ solves the free Heat equation: $\partial_t f - \Delta f = 0$, $f(0) = v$.

# Links between heat/Schrödinger/waves controllability

$\Delta$ is the Laplacian on a bounded $M$ with Dirichlet boundary conditions.

| Controllabilty of | | Restriction on $\Omega$ | Restriction on $T$ |
|---|---|---|---|
| Heat eq. | $\partial_t f - \Delta f = \Omega u$ | *No* | *No* |
| Schrödinger eq. | $i\partial_t \psi - \Delta \psi = \Omega u$ | *Yes* | *No* |
| Wave eq. | $\partial_t^2 w - \Delta w = \Omega u$ | *Yes* | *Yes* |

# Links between heat/Schrödinger/waves controllability

$\Delta$ is the Laplacian on a bounded $M$ with Dirichlet boundary conditions.

| Controllabilty of | | Restriction on $\Omega$ | Restriction on $T$ |
|---|---|---|---|
| Heat eq. | $\partial_t f - \Delta f = \Omega u$ | *No* | *No* |
| Schrödinger eq. | $i\partial_t \psi - \Delta \psi = \Omega u$ | *Yes* | *No* |
| Wave eq. | $\partial_t^2 w - \Delta w = \Omega u$ | *Yes* | *Yes* |

1. $\exists T$, wave control $\Rightarrow \forall T$, heat control
   (by the control transmutation method, cf. Russell, Phung, Miller).
2. $\exists T$, wave control $\Rightarrow \forall T$, Schrödinger control
   (by resolvent conditions, cf. Liu, Miller, Tucsnak-Weiss).
3. $\exists T$, wave control $\Leftrightarrow \exists T$, wave group control: $i\partial_t \psi + \sqrt{-\Delta}\psi = \Omega u$
   (by resolvent conditions, cf. Miller'12)

This leads to the new question : Schrödinger control $\Rightarrow$ heat control ?

# Links between heat/Schrödinger/waves controllability

$\Delta$ is the Laplacian on a bounded $M$ with Dirichlet boundary conditions.

| Controllabilty of | | Restriction on $\Omega$ | Restriction on $T$ |
|---|---|---|---|
| Heat eq. | $\partial_t f - \Delta f = \Omega u$ | *No* | *No* |
| Schrödinger eq. | $i\partial_t \psi - \Delta \psi = \Omega u$ | *Yes* | *No* |
| Wave eq. | $\partial_t^2 w - \Delta w = \Omega u$ | *Yes* | *Yes* |

1. $\exists T$, wave control $\Rightarrow \forall T$, heat control
   (by the control transmutation method, cf. Russell, Phung, Miller).
2. $\exists T$, wave control $\Rightarrow \forall T$, Schrödinger control
   (by resolvent conditions, cf. Liu, Miller, Tucsnak-Weiss).
3. $\exists T$, wave control $\Leftrightarrow \exists T$, wave group control: $i\partial_t \psi + \sqrt{-\Delta}\psi = \Omega u$
   (by resolvent conditions, cf. Miller'12)

This leads to the new question : Schrödinger control $\Rightarrow$ heat control ?
No but: Schrödinger $\Rightarrow$ fractional diffusion $\partial_t f + (-\Delta)^s f = \Omega u$, $s > 1$.

# Abstract semigroup framework: $t \mapsto e^{-tA}$ observed by $C$.

Hilbert spaces $\mathcal{E}$ (states), $\mathcal{F}$ (observations). Semigroup $e^{-tA}$ on $\mathcal{E}$.
Bounded (in this talk) operator $C \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ (defines what is observed).
Its adjoint $C^*$ defines how the input $u : t \mapsto \mathcal{F}$ acts in order to control.

**Example (Heat on the domain $M$ observed on $\Omega \subset M$)**

$A = -\Delta \geqslant 0$, $\mathcal{E} = \mathcal{F} = L^2(M)$, $\mathcal{D}(A) = H^2(M) \cap H^1_0(M)$, $C = \Omega$.

**Null-controllability in time $T$ of $\partial_t f + A^* f = C^* u$, with $u \in L^2(\mathbb{R}; \mathcal{F})$**

$$\forall f(0) \in \mathcal{E},\ \exists u,\ \text{such that} \quad f(T) = 0 \text{ and } \int_0^T \|u(t)\|^2 dt \leqslant \kappa_T \|f(0)\|^2.$$

$$\Updownarrow$$

**Final-observability in time $T$ (at cost $\kappa_T$)**

$$(FinalObs) \quad \|e^{-TA} v\|^2 \leqslant \kappa_T \int_0^T \|C e^{-tA} v\|^2 dt, \quad v \in \mathcal{E}.$$

# Which PDEs belong to this unitary group framework?

Linear PDEs with time-independent coefficients and conservation of "energy" (= some Hilbert norm):

- Wave equation
$$\partial_t^2 w - \Delta w = 0,$$
- Schrödinger equation
$$i\partial_t \psi - \Delta \psi + V\psi = 0,$$
- Plate equation
$$\partial_t^2 z + \Delta^2 z = 0,$$
- Linear elasticity,
- Maxwell,
- ...

Boundary or interior control.

# The unitary group framework: $t \mapsto e^{itA}$ observed by $C$.

Hilbert spaces $\mathcal{E}$ (states), $\mathcal{F}$ (observations). Self-adjoint $A = A^*$.
$(e^{itA})^* = e^{-itA}$, hence unitary $\|e^{itA}u\| = \|u\|$ (conservation law).

**Example (Schrödinger on the domain $M$ observed on $\Omega \subset M$)**

$A = -\Delta \geqslant 0$, $\mathcal{E} = \mathcal{F} = L^2(M)$, $\mathcal{D}(A) = H^2(M) \cap H^1_0(M)$, $C = \Omega$.

**Controllability in time $T$ of $i\partial_t\psi + A^*\psi = C^*u$, with $u \in L^2(\mathbb{R}; \mathcal{F})$**

$\forall v, \psi(0) \in \mathcal{E}$, $\exists u$, such that $\psi(T) = v$ and $\displaystyle\int_0^T \|u(t)\|^2 dt \leqslant \kappa_T \|\psi(0) - v\|^2$.

$\Updownarrow$

**Observability in time $T$ (at cost $\kappa_T$)**

$$(ExactObs) \quad \|v\|^2 = \|e^{iTA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{itA}v\|^2 dt, \quad v \in \mathcal{E}.$$

$t \mapsto \psi(t) = e^{itA}u$ solves the free Schrödinger: $i\partial_t\psi + A\psi = 0$, $\psi(0) = v$.

# Resolvent conditions for control (= a.k.a. Hautus tests)

From spectral to dynamic inequalities by (unitary) Fourier transform on $\mathcal{E}$.

**Recall: Huang-Prüss'84 test for exponential stability of $t \mapsto e^{-tA}$**

$$\|(A-\lambda)^{-1}\| \leqslant m, \quad \operatorname{Re}\lambda < 0.$$

# Resolvent conditions for control (= a.k.a. Hautus tests)

From spectral to dynamic inequalities by (unitary) Fourier transform on $\mathcal{E}$.

**Recall: Huang-Prüss'84 test for exponential stability of $t \mapsto e^{-tA}$**

$$\|(A-\lambda)^{-1}\| \leqslant m, \quad \operatorname{Re}\lambda < 0.$$

**Hautus test for observability of $t \mapsto e^{itA}$, $A = A^*$, by $C$, for some $T$**

$$\|v\|^2 \leqslant m\|(A-\lambda)v\|^2 + \tilde{m}\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}.$$

Zhou-Yamamoto'97 (Huang-Prüss). Burq-Zworski'04 ($\Rightarrow$). Miller'05 ($\Leftrightarrow$):

$$T > \pi\sqrt{m} \quad \text{and} \quad \kappa_T = 2\tilde{m}T/(T^2 - m\pi^2).$$

For part 3, keep in mind: if $m \sim \tilde{m}$, then $T \sim \kappa_T \sim \sqrt{m}$.

**Recall: Observability of $t \mapsto e^{itA}$, $A = A^*$, by $C$ at time $T$ means**

$$(ExactObs) \quad \|v\|^2 = \|e^{iTA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{itA}v\|^2 dt, \quad v \in \mathcal{E}.$$

# Resolvent conditions for control (= a.k.a. Hautus tests)

From spectral to dynamic inequalities by (unitary) Fourier transform on $\mathcal{E}$.

**Recall: Huang-Prüss'84 test for exponential stability of $t \mapsto e^{-tA}$**

$$\|(A-\lambda)^{-1}\| \leqslant m, \quad \operatorname{Re}\lambda < 0.$$

**Hautus test for observability of $t \mapsto e^{itA}$, $A = A^*$, by $C$, for some $T$**

$$\|v\|^2 \leqslant m\|(A-\lambda)v\|^2 + \tilde{m}\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}.$$

Zhou-Yamamoto'97 (Huang-Prüss). Burq-Zworski'04 ($\Rightarrow$). Miller'05 ($\Leftrightarrow$):

$$T > \pi\sqrt{m} \quad \text{and} \quad \kappa_T = 2\tilde{m}T/(T^2 - m\pi^2).$$

For part 3, keep in mind: if $m \sim \tilde{m}$, then $T \sim \kappa_T \sim \sqrt{m}$.

**Similar Hautus test for wave $\ddot{w} + Aw = C^* f$, $A > 0$, for some $T$**

$$\|v\|^2 \leqslant \frac{m}{\lambda}\|(A-\lambda)v\|^2 + \tilde{m}\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}^*.$$

Liu'97 (Huang-Prüss), Miller'05 ($\Leftarrow$), R.T.T.Tucsnak'05, Miller'12.

# Quasimodes disproving controllability

Recall the basic resolvent condition for the wave equation:

$$\|v\|^2 \leqslant \frac{m}{\lambda}\|(A-\lambda)v\|^2 + \tilde{m}\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}^*.$$

**A quasimode for the wave-type equation $\ddot{w} + Aw$, $C \in \mathcal{L}(\mathcal{D}(\sqrt{A}))$ is**

a sequence $(v_\lambda)$ in $\mathcal{D}(A)$, $\lambda \to \infty$, $Cv_\lambda = o(1)$ and $(A-\lambda)v_\lambda = o(\sqrt{\lambda})$.

Basic quasimodes $(A-\lambda)v_\lambda = O(1)$ are enough, like in a rectangle observed from a strip (a tensor product observed on one factor only).

Not for Schrödinger. Need a better Gaussian beams approach (Ralston).

$$\|v\|^2 \leqslant m\|(A-\lambda)v\|^2 + \tilde{m}\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R},$$

**A quasimode for the Schrödinger group $i\dot{\psi} + A\psi$, $C \in \mathcal{L}(\mathcal{D}(A))$ is**

a sequence $(v_\lambda)$ in $\mathcal{D}(A)$, $\lambda \to \lambda_0 \in \overline{\mathbb{R}}$, $Cv_\lambda = o(1)$ and $(A-\lambda)v_\lambda = o(1)$.

# Which $\lambda$ matter when coefficients depend on it ?

Now the coefficients $\tilde{m}$ and $m$ are positive functions on $\mathbb{R}$.

**Resolvent condition with variable coefficients**

$$\|v\|^2 \leqslant m(\lambda)\|(A-\lambda)v\|^2 + \tilde{m}(\lambda)\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}.$$

Can be recovered from its restriction to the spectrum of $A$, with loss:
for $\lambda \notin \sigma(A)$, $\|(A-\lambda)^{-1}\| \leqslant |\mu-\lambda|^{-1}$, where $\mu \in \sigma(A)$ is closest to $\lambda$;
hence $\|(A-\mu)v\| \leqslant \|(A-\lambda)v\| + \|(\lambda-\mu)v\| \leqslant 2\|(A-\lambda)v\|$.
But no loss when restricted to the convex hull of the spectrum :

**Restricted resolvent condition (for simplicity assume $A \geqslant 0$).**

$$\|v\|^2 \leqslant m_\sigma(\lambda)\|(A-\lambda)v\|^2 + \tilde{m}_\sigma(\lambda)\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \geqslant \inf A,$$

implies the resolvent condition above with symmetric continuation:
$m = m_\sigma$ if $\lambda \geqslant \inf A$, else $m(\lambda) = m_\sigma(2\inf A - \lambda)$, the same for $\tilde{m}$. Proof:
$\|(B-\mu)v\| \leqslant \|(B+\mu)v\|$, for $B = A - \inf A \geqslant 0$ and $\mu = \inf A - \lambda > 0$,
since $B - \mu = (B+\mu)\varphi(B/\mu)$, with $\varphi(t) = \frac{t-1}{t+1} \in [-1,1]$ for $t \geqslant 0$.

# Exercise: Laplacian resolvent condition on $M = [a, b]$

$A = -\partial_x^2 \geqslant \inf A = \pi/|M| > 0$, where $|M| = b - a$. $C = \Omega$. The goal is:

$$\|v\|^2 \leqslant \frac{m}{\lambda}\|(\partial_x^2+\lambda)v\|^2+\tilde{m}\|\Omega v\|^2, \quad v \in H^2(M)\cap H_0^1(M), \quad \lambda \geqslant \inf A > 0.$$

Let $\chi \in C^\infty(\mathbb{R})$, $\chi = 1$ out of $\Omega$, $\chi = 0$ near $0 \in \Omega \subset M$ (for simplicity).
Sufficient to prove: $\|\chi v\|^2 \leqslant m\frac{1}{\lambda}\|f\|^2 + \tilde{m}\|\Omega v\|^2$, where $f = (\partial_x^2 + \lambda)v$.

Solve $g = u'' + \lambda u$ is with null initial conditions $u(0) = u'(0) = 0$:
$u(x) = \frac{1}{\sqrt{\lambda}} \int_0^x \sin(\sqrt{\lambda}(x - y))g(y)dy$. Hence $\|u\|^2 \leqslant \frac{|M|^2}{\lambda}\|g\|^2$.

Apply to $u = \chi v$: $g = (\partial_x^2 + \lambda)(\chi v) = \chi f + r$, with $r = 2\chi' v' + \chi'' v$.

- $\chi f$ contributes $\frac{1}{\lambda}\|f\|^2$.
- By integration by parts and $r = 0$ out of $\Omega$: $r$ contributes $\|\Omega v\|^2$.

## Resolvent condition with variable coefficients

$$\|v\|^2 \leqslant m(\lambda)\|(A-\lambda)v\|^2 + \tilde{m}(\lambda)\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}.$$

## Wavepackets condition (Chen-F-N-S'91, Ramdani-T-T-T'05)

$$\|v\|^2 \leqslant \tilde{d}(\lambda)\|Cv\|^2, \quad v \in \mathbf{1}_{|A-\lambda|^2 \leqslant d(\lambda)}\mathcal{E}, \quad \lambda \in \mathbb{R}.$$

The resolvent condition implies the wavepackets condition with any $\tilde{d} > \tilde{m}$ and associated $d = (1 - \frac{\tilde{m}}{\tilde{d}})/m$, for example $\tilde{d} = 2\tilde{m}$ and $d = 1/(2m)$.
Assuming admissibility with coefficients $l$ and $\tilde{l}$, the wavepackets condition implies the resolvent condition with $\tilde{m} = 2\tilde{d}$ and $m = 2l + (1 + 2\tilde{d}\tilde{l})/d$.
If $d$ and $\tilde{d}$ are constant, this implies observability for all $T$ large enough, but the time estimate is not simple !
If moreover $\sigma_{ess}(\mathcal{A}) = \emptyset$ and $|\lambda - \mu| \geqslant \gamma > 0$ for all eigenvalues $\lambda$ and $\mu$, then the wavepacket condition with $d < \gamma^2$ is equivalent to the

## Eigenfunctions condition

$$\|v\|^2 \leqslant d\|Cv\|^2, \quad \text{for all eigenvector } v \text{ of } A.$$

# Control time: $m$ only matters on the essential spectrum

Assume admissibility. Consider a finite set $S$ of eigenvalues of $A$ such that $\mathcal{E}_S = \mathbf{1}_{A\in S}\,\mathcal{E} = \oplus_{\lambda\in S}\ker(A-\lambda)$ has finite dim and $Cv \neq 0$, $0 \neq v \in \mathcal{E}_S$.

**Tucsnak & Weiss'09 (based on simultaneous control T&W'00)**

If the restriction of $A$ to $\mathbf{1}_{A\notin S}\,\mathcal{E} = \mathcal{E}_S^\perp$ is observable by $C$ then so does $A$ with the same observation time $T$.

Assume $A \geqslant 0$ and the restricted resolvent condition with a constant $\tilde{m}_\sigma$:

$$\|v\|^2 \leqslant m_\sigma(\lambda)\|(A-\lambda)v\|^2 + \tilde{m}_\sigma\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \geqslant \inf A.$$

Assume the spectrum of $A$ is only isolated eigenvalues of finite multiplicity. Define the essential coefficient $m_{ess} = \limsup_\infty m_\sigma(\lambda)$.

Then observability of $t \mapsto e^{itA}$ holds for all time $T > T_{ess} := \pi\sqrt{m_{ess}}$.

# Fast observability of high modes at low cost $\kappa_T \sim \frac{const.}{T}$

Application: if $A \geqslant 0$ with compact resolvent satisfies admissibility and the resolvent condition with $m(\lambda) \to 0$ as $\lambda \to +\infty$ and with $\tilde{m}$ bounded, then observability holds for all $T > 0$.

But observability for all $T > 0$ does not imply $m(\lambda) \to 0$ as $\lambda \to +\infty$.

Recall the link of $m$ and $\tilde{m}$ to the time $T$, admissibility $K_T$ and cost $\kappa_T$:

$$T > \pi\sqrt{m}, \ \kappa_T = 2\tilde{m}T/(T^2 - m\pi^2), \quad \text{and} \quad M = T^2\kappa_T K_T, \ \tilde{m} = 2T\kappa_T.$$

If the resolvent condition holds with $m(\lambda) \to 0$ as $\lambda \to +\infty$, then the restriction of $A$ to $\mathbf{1}_{|A|>\lambda}\mathcal{E}$ is observable in time $T(\lambda) = \pi\sqrt{2m(\lambda)}$ at low cost $\kappa_{T(\lambda)} = 4\tilde{m}/T(\lambda)$.

Conversely, if the restriction of $A$ to $\mathbf{1}_{|A|>\lambda}\mathcal{E}$ is observable in time $T(\lambda)$ with $T(\lambda) \to 0$ and $\gamma = \limsup T(\lambda)\kappa_{T(\lambda)} < \infty$ as $\lambda \to +\infty$, then the resolvent condition holds with $m(\lambda) < \gamma K_1 T(\lambda)$ for large $\lambda$. In particular, this implies observability of $A$ for all $T > 0$.

# Links between Wave/Wave group/Schrödinger/Plates

| Wave | | Wave Group |
|---|---|---|
| $\ddot{w} + Aw$ (for some $T$) | $\Leftrightarrow$ | $i\dot{y} + \sqrt{A}y$ (for some $T$) |
| | | $\Downarrow (s > 1)$ |
| $\ddot{z} + A^s z$ (any $T$ if...) | $\Leftrightarrow$ | $i\dot{\psi} + (\sqrt{A})^s \psi$ (any $T$ if $\sigma(A)$ discrete). |
| Plates if $s = 2$ | | Schrödinger if $s = 2$ |

For example: the spectrum $\sigma(A)$ is discrete when $A$ has compact resolvent.
Dirichlet waves and Schrödinger $\rightsquigarrow$ hinged plates $z = \Delta z = 0$ on $\partial M$.
Liu'97 ($s = 2$), Zhou-Yamamoto'97 ($s > 1$): Wave $\Rightarrow$ Plates, without explicit improved resolvent conditions and without information on $T$.

# Scale of resolvent conditions for wave-type equations

(Includes Neumann boundary observation of the Dirichlet Laplacian $-A$.)

Assume $A > 0$. Sobolev spaces: $\mathcal{H}^s = \mathcal{D}(A^{s/2})$, $\|v\|_s = \|A^{s/2}v\|$, $s \in \mathbb{R}$.

$$\ddot{w}(t) + Aw(t) = 0, \quad w(0) = w_0 \in \mathcal{H}^1, \dot{w}(0) = w_1 \in \mathcal{H}^0, \quad \text{obs:}\, Cw(t),$$
$$i\dot{\psi}(t) + A^{s/2}\psi(t) = 0, \quad \psi(0) = \psi_0 \in \mathcal{H}^1, \quad \text{obs:}\, C\psi(t).$$

This is Schrödinger's group $e^{itA}$ for $s = 2$, the wave group $e^{it\sqrt{A}}$ for $s = 1$.

**Observability for the wave $w$ is equivalent to: $\exists s \geqslant 1$ such that**

$$\|v\|_1^2 \leqslant \frac{m_s}{\lambda^{2(1-1/s)}} \|(A^{s/2} - \lambda)v\|_1^2 + \tilde{m}_s \|Cv\|^2, \quad v \in \mathcal{H}^{1+s}, \quad \lambda > 0.$$

This is an "improved resolvent condition" for the observability of $\psi$ since $m(\lambda) := \dfrac{m_s}{\lambda^{2(1-1/s)}} \to 0$ as $\lambda \to \infty$ if $s > 1$. Whereas $m(\lambda) = m_s$ if $s = 1$.

## Resolvent conditions for fractional powers of $A$ (assume admissibility)

The Schrödinger group of $A$ is observable by $C$ if and only if: $\exists s \geqslant 1$,

$$\|v\|^2 \leqslant \frac{m_s}{\lambda^{2(1-1/s)}}\|(A^s-\lambda)v\|^2 + \tilde{m}_s\|Cv\|^2, \quad v \in \mathcal{D}(A^s), \quad \lambda > 0.$$

This clarifies the red link between Wave/Wave group/Schrödinger/Plates:

$$\begin{array}{lcl} \ddot{w} + Aw \text{ (for some } T) & \Leftrightarrow & i\dot{y} + \sqrt{A}y \text{ (for some } T) \\ & & \quad\quad \Downarrow (s > 1) \\ \ddot{z} + A^s z \text{ (any } T \text{ if} \ldots) & \Leftrightarrow & i\dot{\psi} + (\sqrt{A})^s\psi \text{ (any } T \text{ if } \sigma(A) \text{ discrete),} \end{array}$$

To complete the proof of the scale of resolvent conditions for the wave equation, we just have to prove the blue link.

## The wave equation $\ddot{w} + Aw \Leftrightarrow$ the wave group $i\dot{y} + \sqrt{A}y$

The wave equation is the unitary group $e^{itW}$, with $W$ defined on $\mathcal{D}(W) = \mathcal{H}^0 \times \mathcal{H}^1$ by $W(z_0, z_1) = -i(z_1, -Az_0)$ and $C(z_0, z_1) = Cz_0$.
Simplify the resolvent conditions: $\forall v \in \mathcal{D}(W)$, $\forall \lambda \in \mathbb{R}$,

$$\|v\|^2 \leqslant m\|(W-\lambda)v\|^2 + \tilde{m}\|Cv\|^2,$$

using $JWJ^{-1} = \sqrt{A}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with isomorphism $J : \mathcal{H}^1 \times \mathcal{H}^0 \to \mathcal{H}^1 \times \mathcal{H}^1$ defined by $J(z_0, z_1) = (z_0 - iA^{-1/2}z_1, z_0 + iA^{-1/2}z_1)/\sqrt{2}$.

The resulting resolvent condition is: $\forall w_0, w_1 \in \mathcal{H}^2$, $\forall \lambda \in [\sigma](\sqrt{A})$,

$$\|w_0\|_1^2 + \|w_1\|_1^2 \leqslant m\left(\|(\sqrt{A}-\lambda)w_0\|_1^2 + \|(\sqrt{A}+\lambda)w_1\|_1^2\right) + \frac{\tilde{m}}{2}\|C(w_0+w_1)\|^2.$$

Taking $w_1 = 0$ proves the implication $\Rightarrow$.
For the converse, apply the resolvent condition for $\sqrt{A}$ to $u = w_0$, then write $\|Cw_0\|^2 \leqslant 2\|C(w_0+w_1)\|^2 + 2\|Cw_1\|^2$ and $\|Cw_1\| \leqslant \|C\|\|\sqrt{A}w_1\|_1$, and finally simplify by $\inf(\sqrt{A})\|w_1\|_1 \leqslant \|\sqrt{A}w_1\|_1 \leqslant \|(\sqrt{A}+\lambda)w_1\|_1$, $\lambda > 0$.

# Sufficient resolvent conditions for $t \mapsto e^{-tA}$, $A > 0$

Recall (*ExactObs*) for Schrödinger $\dot{\psi} - iA\psi = 0 \Rightarrow$ (*Res*) with $\delta = 1$
$\Rightarrow$ (*Res*) with $\delta = 0 \Leftrightarrow$ (*ExactObs*) for wave $\ddot{w} + Aw = 0$.

**Theorem (Duyckaerts-Miller'12: Main Result)**

*If the resolvent condition with power-law factor : $\exists m > 0$,*

$$(Res) \quad \|v\|^2 \leqslant m\lambda^{\delta} \left( \frac{1}{\lambda} \|(A - \lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

*holds for some $\delta \in [0, 1)$, then observability (FinalObs) holds for all $T > 0$ with the control cost estimate $\kappa_T \leqslant ce^{c/T^{\beta}}$ for $\beta = \frac{1+\delta}{1-\delta}$ and some $c > 0$.*

Here $C$ is bounded, or admissible to some degree (cf. our paper).

# Sufficient resolvent conditions for $t \mapsto e^{-tA}$, $A > 0$

Recall (*ExactObs*) for Schrödinger $\dot{\psi} - iA\psi = 0 \Rightarrow$ (*Res*) with $\delta = 1$
$\Rightarrow$ (*Res*) with $\delta = 0 \Leftrightarrow$ (*ExactObs*) for wave $\ddot{w} + Aw = 0$.

## Theorem (Duyckaerts-Miller'12: Main Result)

*If the resolvent condition with power-law factor : $\exists m > 0$,*

$$(Res) \quad \|v\|^2 \leqslant m\lambda^{\delta}\left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

*holds for some $\delta \in [0,1)$, then observability (FinalObs) holds for all $T > 0$ with the control cost estimate $\kappa_T \leqslant ce^{c/T^{\beta}}$ for $\beta = \frac{1+\delta}{1-\delta}$ and some $c > 0$.*

Here $C$ is bounded, or admissible to some degree (cf. our paper).

## Theorem (Duyckaerts-Miller'12: Schrödinger to heat)

*If (ExactObs) for Schrödinger $t \mapsto e^{itA}$ holds for some $T$, then (FinalObs) for "higher-order" heat $t \mapsto e^{-tA^{\gamma}}$, $\gamma > 1$ holds for all $T$.*

# Application to the control of diffusions in a potential well

$$A = -\Delta + V \text{ on } \mathcal{E} = L^2(\mathbb{R}), \quad \mathcal{D}(A) = \left\{ u \in H^2(\mathbb{R}) \mid Vu \in L^2(\mathbb{R}) \right\}.$$

$$V(x) = x^{2k}, \ k \in \mathbb{N}, \ k > 0. \quad C = \Omega = (-\infty, x_0), \ x_0 \in \mathbb{R}.$$

# Application to the control of diffusions in a potential well

$$A = -\Delta + V \text{ on } \mathcal{E} = L^2(\mathbb{R}), \quad \mathcal{D}(A) = \left\{ u \in H^2(\mathbb{R}) \mid Vu \in L^2(\mathbb{R}) \right\}.$$

$$V(x) = x^{2k}, \ k \in \mathbb{N}, \ k > 0. \quad C = \Omega = (-\infty, x_0), \ x_0 \in \mathbb{R}.$$

**Theorem (Miller at *CPDEA*, IHP'10)**

$$\|v\|^2 \leqslant m\lambda^{1/k} \left( \frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

*and the decay of the first coefficient cannot be improved.*

**Theorem (Duyckaerts-Miller'12)**

*The diffusion in the potential well* $V(x) = x^{2k}$, $k \in \mathbb{N}$, $k > 1$,

$$\partial_t \phi - \partial_x^2 \phi + V\phi = \Omega u, \quad \phi(0) = \phi_0 \in L^2(\mathbb{R}), \ u \in L^2([0,T] \times \mathbb{R}),$$

*is null-controllable in any time, i.e.* $\forall T > 0$, $\forall \phi_0$, $\exists u$ *such that* $\phi(T) = 0$.

# Necessary resolvent conditions for any semigroup $t \mapsto e^{-tA}$

**Example (worst resolvent condition for the Laplacian on a manifold)**

$-A$ is the Laplacian on the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$,
$C = \Omega$ is the complement a neighborhood of the great circle $\{z = 0\}$.
$e_n(x, y, z) = (x + iy)^n$: $(A - \lambda_n)e_n = 0$ and $\exists a > 0$, $\|e_n\| \geqslant a e^{a\sqrt{\lambda_n}} \|Ce_n\|$.

This leads to the *resolvent condition with exponential factor* : $\exists m > 0$,

$$(Res) \quad \|v\|^2 \leqslant m e^{m(\operatorname{Re}\lambda)^{\alpha}} \left( \|(A - \lambda)v\|^2 + \|Cv\|^2 \right), \; v \in \mathcal{D}(A), \; \operatorname{Re}\lambda > 0.$$

# Necessary resolvent conditions for any semigroup $t \mapsto e^{-tA}$

**Example (worst resolvent condition for the Laplacian on a manifold)**

$-A$ is the Laplacian on the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$,
$C = \Omega$ is the complement a neighborhood of the great circle $\{z = 0\}$.
$e_n(x, y, z) = (x + iy)^n$: $(A - \lambda_n)e_n = 0$ and $\exists a > 0$, $\|e_n\| \geqslant ae^{a\sqrt{\lambda_n}}\|Ce_n\|$.

This leads to the *resolvent condition with exponential factor* : $\exists m > 0$,

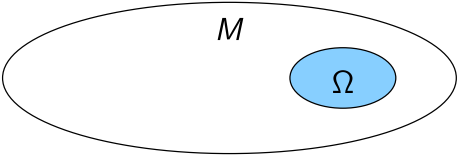
$$(Res) \quad \|v\|^2 \leqslant me^{m(\operatorname{Re}\lambda)^{\alpha}} \left(\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \ v \in \mathcal{D}(A), \ \operatorname{Re}\lambda > 0.$$

**Theorem (Duyckaerts-Miller'12)**

*If (FinalObs) holds for some $T > 0$ then (Res) holds with $\alpha = 1$.*
*If (FinalObs) holds for all $T \in (0, T_0]$ with the control cost $\kappa_T = ce^{c/T^{\beta}}$ for some $\beta > 0$, $c > 0$, $T_0 > 0$, then (Res) holds with $\alpha = \frac{\beta}{\beta+1} < 1$.*

Still valid for $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ admissible, i.e. $\int_0^T \|Ce^{-tA}v\|^2 dt \leqslant k_T\|v\|^2$.

# An observability estimate for sums of eigenfunctions




$$\int_M |w(x)|^2 dx \leqslant \tilde{c} e^{c\sqrt{\lambda}} \int_\Omega |w(x)|^2 dx,$$

for all $\lambda > 0$ and $w = \sum_{\mu \leqslant \lambda} e_\mu,$

where $\begin{cases} -\Delta e_\mu = \mu e_\mu & \text{on } M \\ e_\mu = 0 & \text{on } \partial M \end{cases}.$

Lebeau-Robbiano'95 (Carleman estimates), Lebeau-Jerison'96, Lebeau-Zuazua'98.

# The direct Lebeau-Robbiano strategy

We may write the previous spectral observability estimate concisely with spectral subspaces of the Dirichlet Laplacian $\mathcal{E}_\lambda = \text{Span}_{\mu \leqslant \lambda} e_\mu$:

$$\|w\| \leqslant \tilde{a} e^{a\sqrt{\lambda}} \|\Omega w\|, \quad w \in \mathcal{E}_\lambda, \quad \lambda > 0.$$

More generally $\mathcal{E}_\lambda$ may be defined by some functional calculus. For example, when $A$ is self-adjoint: $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda} \mathcal{E}$.

**Observability on spectral subspaces (with power $\alpha \in (0,1)$)**

$$(SpecObs) \quad \|w\| \leqslant \tilde{a} e^{a\lambda^\alpha} \|Cw\|, \quad w \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0 > 0.$$

$$\Downarrow$$

**Fast final-observability (at cost $\kappa_T \leqslant \tilde{c} e^{c/T^\beta}$, $\beta = \frac{\alpha}{1-\alpha}$)**

$$(FinalObs) \quad \|e^{-TA} v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA} v\|^2 dt, \quad v \in \mathcal{E}, \quad T > 0.$$

# "Dynamic spectral inequality" for the direct L.-R. strategy

## Observability on spectral subspaces (with power $\alpha \in (0,1)$)

$$(SpecObs) \quad \|w\| \leqslant \tilde{a} e^{a\lambda^{\alpha}} \|Cw\|, \quad w \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0 > 0.$$

$$\Downarrow \quad \beta > 0$$

## Dynamic observability on spectral subspaces ($\alpha \in (0,1)$)

$$\|e^{-TA} w\|^2 \leqslant \tilde{a} e^{a\lambda^{\alpha} + b/T^{\beta}} \int_0^T \|Ce^{-tA} w\|^2 dt, \quad w \in \mathcal{E}_\lambda, \quad T > 0, \; \lambda \geqslant \lambda_0.$$

$$\Downarrow \quad \beta = \frac{\alpha}{1-\alpha}$$

## Fast final-observability (at cost $\kappa_T \leqslant \tilde{c} e^{c/T^{\beta}}$, $\beta = \frac{\alpha}{1-\alpha}$)

$$(FinalObs) \quad \|e^{-TA} v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA} v\|^2 dt, \quad v \in \mathcal{E}, \quad T > 0.$$

# Sufficient resolvent conditions for $t \mapsto e^{-tA}$, $A > 0$

Now we are ready to sketch the proof of the main result, which we recall:

**Theorem (Duyckaerts-Miller'12: Main Result)**

*If the resolvent condition with power-law factor : $\exists m > 0$,*

$$(Res) \quad \|v\|^2 \leqslant m\lambda^{\delta} \left( \frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

*holds for some $\delta \in [0,1)$, then observability $(FinalObs)$ holds for all $T > 0$ with the control cost estimate $\kappa_T \leqslant ce^{c/T^{\beta}}$ for $\beta = \frac{1+\delta}{1-\delta}$ and some $c > 0$.*

In this talk, we sketch the proof for $\delta = 1/3$ to simplify the computations. Then $\beta = 2$.

# Sketch of proof of the Main Result (with $\delta = 1/3$)

Recall $A > 0$, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\,\mathcal{E}$ hence $\mathcal{E}_{\lambda^2} = \mathbf{1}_{\sqrt{A}<\lambda}\,\mathcal{E}$.

$$(Res)\quad \|v\|^2 \leqslant m\lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A),\ \lambda > 0.$$

$$\Rightarrow (FinalObs)\quad \|e^{-TA}v\|^2 \leqslant ce^{c/T^2} \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E},\ T > 0.$$

# Sketch of proof of the Main Result (with $\delta = 1/3$)

Recall $A > 0$, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\mathcal{E}$ hence $\mathcal{E}_{\lambda^2} = \mathbf{1}_{\sqrt{A}<\lambda}\mathcal{E}$.

$$(Res)\quad \|v\|^2 \leqslant m\lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A),\ \lambda > 0.$$

$$\Rightarrow\quad \|v\|^2 \leqslant m(\lambda^2)^\delta \left(\|(\sqrt{A}-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(\sqrt{A}),\ \lambda > 0.$$

$$\Rightarrow (FinalObs)\quad \|e^{-TA}v\|^2 \leqslant ce^{c/T^2}\int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E},\ T > 0.$$

# Sketch of proof of the Main Result (with $\delta = 1/3$)

Recall $A > 0$, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\mathcal{E}$ hence $\mathcal{E}_{\lambda^2} = \mathbf{1}_{\sqrt{A}<\lambda}\mathcal{E}$.

$$(Res)\quad \|v\|^2 \leqslant m\lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A),\ \lambda > 0.$$

$$\Rightarrow\quad \|v\|^2 \leqslant m(\lambda^2)^\delta \left(\|(\sqrt{A}-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(\sqrt{A}),\ \lambda > 0.$$

$\Rightarrow$ Controllability of waves on $\mathcal{E}_{\lambda^2} \times \mathcal{E}_{\lambda^2}$ for times and cost $\sim m^{\frac{1}{2}}\lambda^\delta$.

$$\Rightarrow (FinalObs)\quad \|e^{-TA}v\|^2 \leqslant ce^{c/T^2}\int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E},\ T > 0.$$

Recall $A > 0$, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\mathcal{E}$ hence $\mathcal{E}_{\lambda^2} = \mathbf{1}_{\sqrt{A}<\lambda}\mathcal{E}$.

$$(Res)\quad \|v\|^2 \leqslant m\lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A),\ \lambda > 0.$$

$$\Rightarrow\quad \|v\|^2 \leqslant m(\lambda^2)^\delta \left(\|(\sqrt{A}-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(\sqrt{A}),\ \lambda > 0.$$

$\Rightarrow$ Controllability of waves on $\mathcal{E}_{\lambda^2} \times \mathcal{E}_{\lambda^2}$ for times and cost $\sim m^{\frac{1}{2}}\lambda^\delta$.

$\Rightarrow$ Controllability of heat on $\mathcal{E}_{\lambda^2}$ for all $T > 0$ at cost $\sim e^{(\lambda^\delta)^2/T}$.

$$\Rightarrow (FinalObs)\quad \|e^{-TA}v\|^2 \leqslant ce^{c/T^2}\int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E},\ T > 0.$$

# Sketch of proof of the Main Result (with $\delta = 1/3$)

Recall $A > 0$, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\mathcal{E}$ hence $\mathcal{E}_{\lambda^2} = \mathbf{1}_{\sqrt{A}<\lambda}\mathcal{E}$.

$$(Res)\quad \|v\|^2 \leqslant m\lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A),\ \lambda > 0.$$

$$\Rightarrow \|v\|^2 \leqslant m(\lambda^2)^\delta \left(\|(\sqrt{A}-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(\sqrt{A}),\ \lambda > 0.$$

$\Rightarrow$ Controllability of waves on $\mathcal{E}_{\lambda^2} \times \mathcal{E}_{\lambda^2}$ for times and cost $\sim m^{\frac{1}{2}}\lambda^\delta$.

$\Rightarrow$ Controllability of heat on $\mathcal{E}_{\lambda^2}$ for all $T > 0$ at cost $\sim e^{(\lambda^\delta)^2/T}$.

$\Rightarrow$ Controllability of heat on $\mathcal{E}_\lambda$ for all $T > 0$ at cost $\sim e^{\lambda^\delta/T}$,
but $\lambda^{1/3}/T \leqslant \lambda^\alpha + 1/T^\beta$ where $\alpha = 2/3$ and $\beta = 2$ satisfy $\beta = \frac{\alpha}{1-\alpha}$,
hence the direct Lebeau-Robbiano strategy in the previous slide applies.

$$\Rightarrow (FinalObs)\quad \|e^{-TA}v\|^2 \leqslant ce^{c/T^2}\int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E},\ T > 0.$$

# Direct Lebeau-Robbiano strategy: cost improvement

We may write the previous spectral observability estimate concisely with spectral subspaces of the Dirichlet Laplacian $\mathcal{E}_\lambda = \mathrm{Span}_{\mu \leqslant \lambda} e_\mu$:

$$\|w\| \leqslant \tilde{a} e^{a\sqrt{\lambda}} \|\Omega w\|, \quad w \in \mathcal{E}_\lambda, \quad \lambda > 0.$$

More generally $\mathcal{E}_\lambda$ may be defined by some functional calculus.
For example, when $A$ is self-adjoint: $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda} \mathcal{E}$.

**Observability on spectral subspaces (with power $\alpha \in (0,1)$)**

$$(SpecObs) \quad \|w\| \leqslant \tilde{a} e^{a\lambda^{\alpha}} \|Cw\|, \quad w \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0 > 0.$$

$$\Downarrow$$

**Fast final-observability (at cost $\kappa_T \leqslant \tilde{c} e^{c/T^{\beta}}$, $\beta = \frac{\alpha}{1-\alpha}$)**

$$(FinalObs) \quad \|e^{-TA} v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA} v\|^2 dt, \quad v \in \mathcal{E}, \quad T > 0.$$

# Sketch of proof in 3 slides: $(SpecObs) \Rightarrow (FinalObs)$.

The goal is fast final-observability (with $\kappa_T \leqslant ce^{c/T}$):

$$(FinalObs) \qquad \|e^{T\Delta}v\|^2 \leqslant \kappa_T \int_0^T \|\Omega e^{t\Delta}v\|^2 dt, \qquad v \in \mathcal{E}, \quad T > 0.$$

Aim at final-observability with approximation rate $\varepsilon_\tau \to 0$ as $\tau \to 0$:

$$(ApproxObs) \qquad \frac{1}{\kappa_\tau}\|e^{\tau\Delta}v\|^2 - \frac{\varepsilon_\tau}{\kappa_\tau}\|v\|^2 \leqslant \int_0^\tau \|\Omega e^{t\Delta}v\|^2 dt, \qquad v \in \mathcal{E},$$

Assume observability on spectral subspaces ($\|\Omega\| \leqslant 1$, forget constants!):

$$(SpecObs) \qquad \|w\|^2 \leqslant e^{\lambda}\|\Omega w\|^2, \qquad \lambda > 0, \quad w \in \mathcal{E}_{\lambda^2}.$$

Hope to compensate this bad spatial cost on $\mathcal{E}_{\lambda^2}$ with the good dissipation in time on the remaining state space $\mathcal{E}_{\lambda^2}^{\perp}$ : take $\lambda \sim 1/\tau$, $v = w + Pv$ ?

Spatial cost : $\quad \tau\|e^{\tau\Delta}w\|^2 \leqslant e^{\lambda} \int_0^\tau \|\Omega e^{t\Delta}w\|^2 dt, \quad w \in \mathcal{E}_{\lambda^2}.$

Time decay: $\quad \|Pe^{\tau\Delta}v\| \leqslant e^{-\tau\lambda^2}\|v\|, \quad v \in \mathcal{E}, \quad P =$Projection on $\mathcal{E}_{\lambda^2}^{\perp}$.

Difficulty: observation of $u$ is not negligible since decay needs time.
Cure: do not observe too early. Observe only on $[\tau/2, \tau]$: $e^{\frac{\tau}{2}\Delta}v = w + u$,

$$\begin{cases} \frac{\tau}{2}e^{-\lambda}\|e^{\frac{\tau}{2}\Delta}w\|^2 \leqslant \int_0^{\frac{\tau}{2}} \|\Omega e^{t\Delta}w\|^2 dt, & w \in \mathcal{E}_{\lambda^2}, \\ \int_0^{\frac{\tau}{2}} \|\Omega e^{t\Delta}u\|^2 dt \leqslant \frac{\tau}{2}\|u\|^2 \leqslant \frac{\tau}{2}e^{-\tau\lambda^2}\|v\|^2, & u = Pe^{\frac{\tau}{2}\Delta}v \perp \mathcal{E}_{\lambda^2}. \end{cases}$$

Use $e^{\tau\Delta}v = e^{\frac{\tau}{2}\Delta}w + e^{\frac{\tau}{2}\Delta}u$ and $\Omega e^{t\Delta}w = \Omega e^{(\frac{\tau}{2}+t)\Delta}v - \Omega e^{t\Delta}u$:

$$\frac{\tau}{8}e^{-\lambda}\|e^{\tau\Delta}v\|^2 - \frac{\tau}{4}e^{-\lambda}\|e^{\frac{\tau}{2}\Delta}u\|^2 \leqslant \frac{\tau}{4}e^{-\lambda}\|e^{\frac{\tau}{2}\Delta}w\|^2 \leqslant \int_{\frac{\tau}{2}}^{\tau} \|\Omega e^{t\Delta}v\|^2 dt + \frac{\tau}{2}\|u\|^2$$

Use $\|e^{\frac{\tau}{2}\Delta}u\|^2 \leqslant \|u\|^2 \leqslant e^{-\tau\lambda^2}\|v\|^2$ and $\frac{\tau}{2} + \frac{\tau}{4}e^{-\lambda} < \tau < \frac{1}{8\tau} < \frac{1}{8}e^{1/\tau}$:

$$\frac{1}{8}e^{-\lambda-1/\tau}\|e^{\tau\Delta}v\|^2 - \frac{1}{8}e^{-\tau\lambda^2+1/\tau}\|v\|^2 \leqslant \int_{\frac{\tau}{2}}^{\tau} \|\Omega e^{t\Delta}v\|^2 dt.$$

Plug $\lambda = \gamma/\tau$, $f(\tau) = \frac{1}{8}e^{-\lambda-1/\tau} = \frac{1}{8}e^{-(\gamma+1)/\tau}$, take $q = \frac{1}{\gamma-1}$ so that:

$$f(\tau)\|e^{\tau\Delta}v\|^2 - f(q\tau)\|v\|^2 \leqslant \int_0^{\tau} \|\Omega e^{t\Delta}v\|^2 dt, \quad v \in \mathcal{E},\ \tau \in (0, \frac{1}{4}).$$

Recall our goal (FinalObs) $\|e^{T\Delta}v\|^2 \leqslant \kappa_T \int_0^T \|\Omega e^{t\Delta}v\|^2 dt$, $\kappa_T \leqslant ce^{c/T}$.

Our choice $\lambda = \gamma/\tau$, e.g. $\gamma = 3$, yields $f(\tau) = \frac{1}{8}e^{-4/\tau}$, $q = 1/2 < 1$, and (ApproxObs) with approximation rate $\varepsilon_\tau = 8f(\tau/(\gamma-2)) \to 0$ as $\tau \to 0$:

$$f(\tau)\|e^{\tau\Delta}v\|^2 - f(q\tau)\|v\|^2 \leqslant \int_0^\tau \|\Omega e^{t\Delta}v\|^2 dt, \quad v \in \mathcal{E},\ \tau \in (0, \frac{1}{4}).$$

As Lebeau and Robbiano, write $T$ as a geometric sequence of times $\tau$ with ratio $q$, i.e. partition $(0, T] = \cup(T_{k+1}, T_k]$ with $T_{k+1} - T_k = \tau_k = q\tau_{k-1}$, $k \in \mathbb{N}$, hence $T = T_0 = \sum_k \tau_k = \tau_0/(1-q)$.
Applying (ApproxObs) on $(T_{k+1}, T_k]$ and adding the telescoping series:

$$f(\tau_0)\|e^{T_0\Delta}v\|^2 - 0 \times \|v\|^2 \leqslant \int_0^{T_0} \|\Omega e^{t\Delta}v\|^2 dt.$$

Our goal is achieved: $\kappa_T \leqslant \dfrac{1}{f(\tau_0)} = \dfrac{1}{f((1-q)T)} = 8e^{\frac{\gamma^2+1}{\gamma-2}\frac{1}{T}} = 8e^{10/T}$.

# Direct Lebeau-Robbiano strategy: perturbation framework

Hilbert spaces $\mathcal{E}$ (states), $\mathcal{F}$ (observations). Semigroup $e^{-tA}$ on $\mathcal{E}$.
*Observation operator* $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ with admissibility condition

$$\int_0^T \|Ce^{-tA}v\|^2 dt \leqslant \mathrm{K}_T \|v\|^2, \quad v \in \mathcal{D}(A), \quad T > 0.$$

*Reference observation operator* $C_0$ with final-observability property:

$$\|e^{-TA}v\|^2 \leqslant b_0 e^{2T\psi(1/T)} \int_0^T \|C_0 e^{-tA}v\|^2 dt, \quad v \in \mathcal{D}(A), \quad T \in (0, T_0).$$

*Scale of spaces* $\mathcal{E}_\lambda \subset \mathcal{E}_\mu \subset \mathcal{E}$, $\lambda \geqslant \mu \geqslant \lambda_0$ with semigroup growth property:

$$\|P_\lambda e^{-tA}v\| \leqslant m_0 e^{m\lambda/\varphi(\lambda)} e^{-\lambda t}\|v\|, \quad v \in \mathcal{D}(A), \quad t \in (0, T_0), \quad \lambda \geqslant \lambda_0.$$

$P_\lambda =$Projection on $\mathcal{E}_\lambda^\perp$. Observability on growth subspaces $\mathcal{E}_\lambda$ w.r.t. $C_0$:

$$(SpecObs) \quad \|C_0 w\|^2 \leqslant a_0 e^{2\lambda/\varphi(\lambda)} \|Cw\|^2, \quad w \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0.$$

If $s \mapsto 1/\psi^{-1}\left(\frac{\varphi(q^s)}{p}\right) = \tau(q^s)$ is integrable at $+\infty$ for some $p > m + 1$ and $q > 1$, then final-observability (FinalObs) holds for all $T > 0$.

# Direct Lebeau-Robbiano strategy: dynamic condition

Hilbert spaces $\mathcal{E}$ (states), $\mathcal{F}$ (observations). Semigroup $e^{-tA}$ on $\mathcal{E}$.
*Observation operator* $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ with admissibility condition

$$\int_0^T \|Ce^{-tA}v\|^2 dt \leqslant \mathrm{K}_T \|v\|^2, \quad v \in \mathcal{D}(A), \quad T > 0.$$

*Scale of spaces* $\mathcal{E}_\lambda \subset \mathcal{E}_\mu \subset \mathcal{E}$, $\lambda \geqslant \mu \geqslant \lambda_0$ with semigroup growth property:

$$\|P_\lambda e^{-tA}v\| \leqslant m_0 e^{m\lambda/\varphi(\lambda)} e^{-\lambda t}\|v\|, \quad v \in \mathcal{D}(A), \quad t \in (0, T_0), \quad \lambda \geqslant \lambda_0.$$

$P_\lambda$ =Projection on $\mathcal{E}_\lambda^\perp$. Dynamic observability on growth subspaces $\mathcal{E}_\lambda$:

$$\|e^{-TA}w\|^2 \leqslant c_0 e^{2T\psi(1/T)\lambda/\varphi(\lambda)} \int_0^T \|Ce^{-tA}w\|^2 dt,$$

$$w \in \mathcal{E}_\lambda, \quad T \in (0, T_0), \quad \lambda \geqslant \lambda_0.$$

If $s \mapsto 1/\psi^{-1}\left(\frac{\varphi(q^s)}{p}\right) = \tau(q^s)$ is integrable at $+\infty$ for some $p > m+1$ and $q > 1$, then final-observability (FinalObs) holds for all $T > 0$.

# Direct Lebeau-Robbiano Strategy: log-improvement

Here $A$ is self-adjoint, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\,\mathcal{E}$, $C$ is bounded or admissible.

**Theorem (Duyckaerts-Miller'12: logarithmic L.-R. strategy)**

*Logarithmic observability on spectral subspaces with* $\alpha > 2$

$$\|v\|^2 \leqslant a e^{a\lambda/((\log(\log\lambda))^\alpha \log\lambda)} \|Cv\|^2, \quad v \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0 > e.$$

$$\Downarrow$$

$$(FinalObs) \quad \|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}, \quad T > 0.$$

# Direct Lebeau-Robbiano Strategy: log-improvement

Here $A$ is self-adjoint, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\mathcal{E}$, $C$ is bounded or admissible.

## Theorem (Duyckaerts-Miller'12: logarithmic L.-R. strategy)

*Logarithmic observability on spectral subspaces with $\alpha > 2$*

$$\|v\|^2 \leqslant a e^{a\lambda/((\log(\log\lambda))^\alpha \log\lambda)} \|Cv\|^2, \quad v \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0 > e.$$

$$\Downarrow$$

$$(FinalObs) \quad \|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}, \quad T > 0.$$

## Theorem (Duyckaerts-Miller'12: logarithmic anomalous diffusion)

*Let* $\varphi(\lambda) = (\log\lambda)^\alpha$, $\alpha > 1$ *or* $\varphi(\lambda) = (\log(\log\lambda))^\alpha \log\lambda$, $\alpha > 2$.
*The following anomalous diffusion is null-controllable in any time* $T > 0$*:*

$$\partial_t\phi + \sqrt{-\Delta}\,\varphi(\sqrt{-\Delta})\phi = \Omega u, \quad \phi(0) = \phi_0 \in L^2(M), \; u \in L^2([0,T]\times M).$$

# Sufficient resolvent conditions for $t \mapsto e^{-tA}$, $A > 0$

Improvement of $\lambda^\delta$, $\delta < 1$, into $\lambda/(\varphi(\lambda))^2$, $\varphi(\lambda) = (\log(\lambda+1))^\alpha$, $\alpha > 1$.

**Theorem (Duyckaerts-Miller'12: Main Result, log-improved)**

*If the resolvent condition with logarithmic factor :* $\exists m > 0$,

$$\|v\|^2 \leqslant \frac{m\lambda}{(\varphi(\lambda))^2}\left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

*holds for some* $\alpha > 1$, *then observability* $(FinalObs)$ *holds for all* $T > 0$.

Here $C$ is bounded, or admissible for the wave equation $\ddot{w} + Aw = 0$.

**Theorem (Duyckaerts-Miller'12: Schrödinger to heat, log-improved)**

*If* $(ExactObs)$ *for Schrödinger* $t \mapsto e^{itA}$ *holds for some* $T$,
*then* $(FinalObs)$ *for "higher-order" heat* $t \mapsto e^{-tA\varphi(A)}$, $\alpha > 1$, $T > 0$.

# The harmonic oscillator observed from a half-line $\Omega \subset \mathbb{R}$

Disproves : controllability of Schrödinger eq. $\Rightarrow$ controllability of heat eq.

$$\partial_t \phi - \partial_x^2 \phi + x^2 \phi = \Omega u \quad \text{versus} \quad i\partial_t \psi - \partial_x^2 \psi + x^2 \psi = \Omega u$$

Here $\Omega = (-\infty, x_0)$, $x_0 \in \mathbb{R}$, and $A = -\partial_x^2 + x^2$ on $\mathcal{E} = L^2(\mathbb{R}) = \mathcal{F}$.

# The harmonic oscillator observed from a half-line $\Omega \subset \mathbb{R}$

Disproves : controllability of Schrödinger eq. $\Rightarrow$ controllability of heat eq.

$$\partial_t \phi - \partial_x^2 \phi + x^2 \phi = \Omega u \quad \text{versus} \quad i\partial_t \psi - \partial_x^2 \psi + x^2 \psi = \Omega u$$

Here $\Omega = (-\infty, x_0)$, $x_0 \in \mathbb{R}$, and $A = -\partial_x^2 + x^2$ on $\mathcal{E} = L^2(\mathbb{R}) = \mathcal{F}$.

**Theorem (Miller at *CPDEA*, IHP'10)**

*Observability (FinalObs) for heat $t \mapsto e^{-tA}$ does not hold for any time.*
*Observability (ExactObs) for Schrödinger $t \mapsto e^{itA}$ holds for some time.*

Eigenvalues are $\lambda_n = 2n + 1$. N.b. $\sum \frac{1}{\lambda_n} = +\infty$ but $\dim \mathcal{F} \neq 1$.
Eigenfunctions $e_n$ are $e_n(x) = c_n (\partial_x - x)^n e^{-x^2/2} = c_n H_n(x) e^{-x^2/2}$, where $c_n = (\sqrt{\pi} 2^n (n!))^{-1/2}$, $H_n = (-1)^n e^{x^2} \partial_x^n e^{-x^2}$ are the Hermite polynomials.

# Sketch of proof 1: non-observability for the heat semigroup

Harmonic oscillator $A = -\partial_x^2 + x^2$ observed from a half line $\Omega = (-\infty, x_0)$.

Disprove (*FinalObs*) $\displaystyle\int_{-\infty}^{+\infty} |(e^{-TA}v)(x)|^2 dx \leqslant \kappa_T^2 \int_0^T \int_{-\infty}^{x_0} |(e^{-tA}v)(x)|^2 dx dt,$

by taking the Dirac mass at $y \notin \Omega$ as initial data $v$ and letting $y \to \infty$.

# Sketch of proof 1: non-observability for the heat semigroup

Harmonic oscillator $A = -\partial_x^2 + x^2$ observed from a half line $\Omega = (-\infty, x_0)$.

Disprove $(FinalObs)$ $\displaystyle\int_{-\infty}^{+\infty} |(e^{-TA}v)(x)|^2 dx \leqslant \kappa_T^2 \int_0^T \int_{-\infty}^{x_0} |(e^{-tA}v)(x)|^2 dxdt,$

by taking the Dirac mass at $y \notin \Omega$ as initial data $v$ and letting $y \to \infty$.

More precisely, $v(x) = e^{-\varepsilon A}(x, y)$, where $\varepsilon$ is a small time, $e^{-tA}(x, y)$ is the kernel of the operator $e^{tA}$. Hence $(e^{-tA}v)(x) = e^{-(t+\varepsilon)A}(x, y)$.

## Bound from below the fundamental state $e_0$ hence the final state

$$\|e^{-TA}v\| \geqslant e^{-(T+\varepsilon)\lambda_0} |e_0(y)| \geqslant c_T \exp\left(-\frac{y^2}{2}\right).$$

## Bound from above the kernel hence the observation: Mehler formula

$$e^{-tA}(x, y) = \frac{e^{-t}}{\sqrt{\pi(1 - e^{-4t})}} \exp\left(-\frac{1 + e^{-4t}}{1 - e^{-4t}} \frac{x^2 + y^2}{2} + \frac{2e^{-2t}}{1 - e^{-4t}} xy\right).$$

# Sketch of proof 2: observability for the Schrödinger group

Harmonic oscillator $A = -\partial_x^2 + x^2$ observed from a half line $\Omega = (-\infty, x_0)$.

Prove (*Res*) using a semiclassical reduction and microlocal propagation.

# Sketch of proof 2: observability for the Schrödinger group

Harmonic oscillator $A = -\partial_x^2 + x^2$ observed from a half line $\Omega = (-\infty, x_0)$.

Prove $(Res)$ using a semiclassical reduction and microlocal propagation.

By the change of variable $u(y) = v(x)$, $y = \sqrt{h}x$, $h = 1/\lambda$,

$$(Res) \quad \|v\|^2 \leqslant m\|(A-\lambda)v\|^2 + m\|\Omega v\|^2, \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

reduces to the semiclassical resolvent condition

$$\int_{-\infty}^{+\infty} |u(y)|^2 dy \leqslant \frac{m}{h^2} \int_{-\infty}^{+\infty} |-h^2 u''(y) + (y^2 - 1)u(y)|^2 dy + m \int_{-\infty}^{\sqrt{h}x_0} |u(y)|^2 dy, \quad u \in C_0^\infty(\mathbb{R}), \quad h \in (0,1].$$

# Sketch of proof 2: observability for the Schrödinger group

Harmonic oscillator $A = -\partial_x^2 + x^2$ observed from a half line $\Omega = (-\infty, x_0)$.

Prove $(Res)$ using a semiclassical reduction and microlocal propagation.

By the change of variable $u(y) = v(x)$, $y = \sqrt{h}x$, $h = 1/\lambda$,

$$(Res) \quad \|v\|^2 \leqslant m\|(A-\lambda)v\|^2 + m\|\Omega v\|^2, \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

reduces to the semiclassical resolvent condition

$$\int_{-\infty}^{+\infty} |u(y)|^2 dy \leqslant \frac{m}{h^2} \int_{-\infty}^{+\infty} |-h^2 u''(y) + (y^2-1)u(y)|^2 dy + m \int_{-\infty}^{\sqrt{h}x_0} |u(y)|^2 dy, \quad u \in C_0^\infty(\mathbb{R}), \quad h \in (0,1].$$

Arguing by contradiction, introduce a semiclassical measure (= Wigner measure) in phase space $(x,\xi) \in \mathbb{R}^2$: it is supported on $\{x^2+\xi^2=1\}$, invariant by rotation and supported in $\{x \geqslant 0\}$.