Internal controllability of parabolic equations coupled by first order terms

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Workshop on control, inverse problems and stabilization of infinite dimensional systems

Marrakesh

Setting

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ \partial_t y_2 = \Delta y_2 + \partial_{x_1} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases}$$

where $y^0 \in L^2(\Omega)^2$, $u \in L^2(\omega \times (0,T))$ and $a \in L^\infty(Q_T)$.

We will be interesting with

- the **null controllability** at time T, i.e.

$$\forall y^0,\ y^T \in L^2(\Omega), \varepsilon > 0\ \exists u \in L^2(Q_T) \text{ s.t. } \|y(T) - y^T\|_{L^2(\Omega)} \leqslant \varepsilon,$$

- the **approximate controllability** at time T, i.e.

$$\forall y^0 \in L^2(\Omega)\ \exists u \in L^2(Q_T) \text{ s.t. } y(T) = 0.$$

Remark : null controllability $\Longrightarrow$ approximate controllability

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Outline

Outline

Boundary condition

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

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Theorem (A. Benabdallah, M. Cristofol, P. Gaitan, L. De Teresa, 2014)

Suppose that $a \in \mathcal{C}^4(\overline{Q}_T)$ and

$$\begin{cases} \exists \gamma \neq \varnothing \text{ an open set of } \partial\omega \cap \partial\Omega, \\ \exists x_0 \in \gamma \text{ s.t. } e_1 \cdot \nu(x_0) \neq 0, \end{cases}$$

where ν is the exterior unitary normal of $\partial\Omega$.
Then system (1) is **null controllable** at time T.

Constant case

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ \partial_t y_2 = \Delta y_2 + \partial_{x_1} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases}$$

Theorem (M. D., P. Lissy, 2016)

Suppose that a is constant.
Then system (1) is **null controllable**.

Positive result

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + \partial_{x_1} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases}$$

Theorem (M. D. , P. Lissy, submitted)

Let $a \in \mathcal{C}^1(\overline{Q}_T)$. If there exists $(x_0, t_0) \in \omega \times (0, T)$ s.t.

$$(\partial_{x_1} a)(x_0, t_0) \neq 0,$$

then system (1) is **null controllable** at time T.

Negative result

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + \partial_{x_1} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases}$$

Theorem (M. D., P. Lissy, submitted)

There exists $a \in \mathcal{C}^\infty([0,\pi])$ such that :

- There exists $(a,b) \subset\subset (0,\pi)$ s.t., for all $T > 0$, system (1) is **null controllable** (hence approximatively controllable) at time T.
- There exists $(a,b) \subset\subset (0,\pi)$ such that, for all $T > 0$, system (1) is **not approximatively controllable** (hence not null controllable) at time T.

Outline

The notion of the algebraic resolvability can be found in :

[1] M. Gromov, *Partial differential relations*, Springer-Verlag , 1986.

And was used for the first time in the control theory of pde's in :

[2] J.-M. Coron & P. Lissy, *Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components*, Invent. Math. **198** (2014), 833–880.

Let $f \in \mathcal{C}_c^\infty([0,1])$. Consider the problem

$$\left\{ \begin{array}{l} \text{Find } (w_1, w_2) \in \mathcal{C}_c^\infty([0,1];\mathbb{R}^2) \text{ s. t. :} \\ a_1 w_1 - a_2 \partial_x w_1 + a_3 \partial_{xx} w_1 + b_1 w_2 - b_2 \partial_x w_2 + b_3 \partial_{xx} w_2 = f, \end{array} \right. \tag{1}$$

with $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.

Problem (1) can be rewritten as follows :

$$\mathcal{L} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = f,$$

where

$$\mathcal{L} := (a_1 - a_2\partial_x + a_3\partial_{xx}, b_1 - b_2\partial_x + b_3\partial_{xx}) .$$

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with $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.

Problem (1) can be rewritten as follows :

$$\mathcal{L}\left(\begin{array}{c} w_1 \\ w_2 \end{array}\right) = f,$$

where

$$\mathcal{L} := \left(a_1 - a_2\partial_x + a_3\partial_{xx}, b_1 - b_2\partial_x + b_3\partial_{xx}\right).$$

If there exists a differential operator $\mathcal{M} := \sum_{i=0}^{M} m_i \partial_x^i$ with $m_0, ..., m_M \in \mathbb{R}$, $M \in \mathbb{N}$ such that

$$\mathcal{L} \circ \mathcal{M} = Id,$$

then $(w_1, w_2) := \mathcal{M} f$ is a solution to Problem (1).

The last equality is formally equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = Id,$$

with

$$\mathcal{L}^*(\varphi) = \begin{pmatrix} a_1 + a_2\partial_x + a_3\partial_{xx} \\ b_1 + b_2\partial_x + b_3\partial_{xx} \end{pmatrix} \varphi.$$

Consider the operator defined for all $\varphi \in \mathcal{C}_c^\infty([0,1])$ by

$$\begin{pmatrix} \mathcal{L}_1^* \\ \mathcal{L}_2^* \\ \partial_x \mathcal{L}_1^* \\ \partial_x \mathcal{L}_2^* \end{pmatrix} \varphi := \begin{pmatrix} a_1 + a_2\partial_x + a_3\partial_{xx} \\ b_1 + b_2\partial_x + b_3\partial_{xx} \\ a_1\partial_x + a_2\partial_{xx} + a_3\partial_{xxx} \\ b_1\partial_x + b_2\partial_{xx} + b_3\partial_{xxx} \end{pmatrix} \varphi = C \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_{xx}\varphi \\ \partial_{xxx}\varphi \end{pmatrix},$$

where

$$C := \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Let us suppose that C is invertible, denote by

$$C^{-1} := \begin{pmatrix} \widetilde{c}_{11} & \widetilde{c}_{12} & \widetilde{c}_{13} & \widetilde{c}_{14} \\ \widetilde{c}_{21} & \widetilde{c}_{22} & \widetilde{c}_{23} & \widetilde{c}_{24} \\ \widetilde{c}_{31} & \widetilde{c}_{32} & \widetilde{c}_{33} & \widetilde{c}_{34} \\ \widetilde{c}_{41} & \widetilde{c}_{42} & \widetilde{c}_{43} & \widetilde{c}_{44} \end{pmatrix}.$$

Then we have

$$C^{-1} \begin{pmatrix} \mathcal{L}_1^* \\ \mathcal{L}_2^* \\ \partial_x \mathcal{L}_1^* \\ \partial_x \mathcal{L}_2^* \end{pmatrix} \varphi = \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_{xx} \varphi \\ \partial_{xxx} \varphi \end{pmatrix},$$

where the first line is given by

$$\widetilde{c}_{11} \mathcal{L}_1^* \varphi + \widetilde{c}_{12} \mathcal{L}_2^* \varphi + \widetilde{c}_{13} \partial_x \mathcal{L}_1^* \varphi + \widetilde{c}_{14} \partial_x \mathcal{L}_2^* \varphi = \varphi.$$

Thus the problem is solved if we define $\mathcal{M}^*$ for all $(\psi_1, \psi_2) \in \mathcal{C}_c^\infty([0,1]; \mathbb{R}^2)$ by

$$\mathcal{M}^* \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \widetilde{c}_{11} \psi_1 + \widetilde{c}_{12} \psi_2 + \widetilde{c}_{13} \partial_x \psi_1 + \widetilde{c}_{14} \partial_x \psi_2.$$

Outline

1. **Presentation of the results**
2. **Algebraic resolvability**
   - **The method**
   - **Efficiency**
3. **Proof of the results**
4. **Conclusion**

Kalman condition

Consider the system of n linear parabolic equations controlled by m controls

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \qquad (2)$$

where $y^0 \in L^2(\Omega;\mathbb{R}^n)$, $u \in L^2(Q_T;\mathbb{R}^m)$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$.

Theorem (F. Ammar Khodja, A. Benabdallah, C. Dupaix, M. González-Burgos, 2009)

System (2) is **null controllable** at time T if and only if

$$\text{Rank}[A|B] = n,$$

where $[A|B] := (B|AB|...|A^{n-1}B)$.

Proof

Fictitious control method :

Consider a solution $(\widehat{y}, \widehat{u})$ to such that

$$\begin{cases} \partial_t \widehat{y} = \Delta \widehat{y} + A\widehat{y} + \widehat{u} & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot,0) = y^0,\ \widehat{y}(\cdot,T) = 0 & \text{in } \Omega, \\ \mathrm{Supp}(\widehat{u}) \subset\subset \omega \times (0,T). \end{cases}$$

If we can find a solution $(\widehat{z}, \widehat{v})$

$$\begin{cases} \partial_t z = \Delta z + Az + Bv + \widehat{u} \ \text{ in } \ q_T, \\ \mathrm{Supp}(z,v) \subset\subset \omega \times (0,T). \end{cases} \qquad \text{(S1)}$$

Then $(y,u) := (\widehat{y} - z, -v)$ will be a solution to the initial system.

Resolution of the algebraic problem

For $f := \mathbb{1}_\omega \widehat{u}$, (S1) writes

$$\mathcal{L}(z,v) = f$$

where

$$\mathcal{L}(z,v) := \quad \partial_t z - \Delta z - Az - Bv.$$

It suffice to find a differential operator $\mathcal{M}$ s.t.

$$\mathcal{L} \circ \mathcal{M} = Id. \quad \text{(S2)}$$

Thus $(z,v) := \mathcal{M}\widehat{u}$ will be a solution to (S1).

Remark : $\widehat{u}$ have to be regular enough

System (S2) is formally equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = Id.$$

Operator $\mathcal{L}^*$ writes :

$$\mathcal{L}^*\varphi = \begin{pmatrix} -\partial_t\varphi - \Delta\varphi - A^*\varphi \\ -B^*\varphi \end{pmatrix}.$$

Let $\mathcal{S} := (\mathcal{S}_1, ..., \mathcal{S}_n)$ given for all (x_1, x_2) by

$$\begin{cases} \mathcal{S}_1(x_1, x_2) := -x_2, \\ \mathcal{S}_2(x_1, x_2) := (\partial_t + \Delta)x_2 - B^*x_1, \\ \mathcal{S}_k(x_1, x_2) := (\partial_t + \Delta)\mathcal{S}_{k-1}(x_1, x_2) - B^*(A^*)^{k-2}x_1, \quad \forall\, k \in \{3, ..., n\}. \end{cases}$$

Then, we obtain

$$\mathcal{S} \circ \mathcal{L}^*\varphi = \begin{pmatrix} B^*\varphi \\ \vdots \\ B^*(A^*)^{n-1}\varphi \end{pmatrix}.$$

Since the rank of $K := (B|AB|...|A^{n-1}B)$ is equal to n, there exists $L \in \mathcal{M}_{n,nm}(\mathbb{R})$ such that $LK^* = I_n$. The operator

$$\mathcal{M} := \mathcal{S}^*L^*$$

is of order 2 in space and 1 in time and is a solution of our problem.

Regular control

Proposition

For all initial condition $y^0 \in L^2(\Omega;\mathbb{R}^n)$, there exists a control $\widehat{u} \in L^2(Q_T;\mathbb{R}^n)$ such that the solution to

$$\begin{cases} \partial_t \widehat{y} = \Delta \widehat{y} + A\widehat{y} + \mathbb{1}_\omega \widehat{u} & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot,0) = y^0,\ \widehat{y}(\cdot,T) = 0 & \text{in } \Omega, \end{cases}$$

is null at time T. Moreover, we have $\mathrm{Supp}(\widehat{u}) \subset \omega \times (\varepsilon, T-\varepsilon)$ and

$$\|\widehat{u}\|_{W_2^{2,1}(Q_T;\mathbb{R}^n)} \leqslant e^{C(1+T+1/T)} \|y^0\|_{L^2(\Omega;\mathbb{R}^n)}.$$

This theorem can be proved using :

1. Method by fictitious control
2. Carleman inequalities

Outline

1. **Presentation of the results**
2. **Algebraic resolvability**
3. **Proof of the results**
   - **Positive result**
   - **Negative result**
4. **Conclusion**

Positive result

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + \partial_{x_1} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases}$$

Theorem (M. D. , P. Lissy, submitted)

Let $a \in \mathcal{C}^1(\overline{Q}_T)$. If there exists $(x_0, t_0) \in \omega \times (0, T)$ s.t.

$$(\partial_{x_1} a)(x_0, t_0) \neq 0,$$

then system (1) is **null controllable** at time T.

Proof : first approach

Fictitious control method :

Let $\widehat{\omega} \subset\subset \omega$. Consider $(\widehat{y}, \widehat{u})$ a solution to

$$\begin{cases} \partial_t \widehat{y}_1 = \Delta \widehat{y}_1 + \hat{u}_1 & \text{in } Q_T, \\ \partial_t \widehat{y}_2 = \Delta \widehat{y}_2 + \partial_{x_1} \widehat{y}_1 + a\widehat{y}_2 + \hat{u}_2 & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(\cdot) = y^0,\ y(\cdot, T) = 0 & \text{in } \Omega, \\ \text{Supp}(\widehat{u}) \subset\subset \widehat{\omega} \times (0,T). \end{cases}$$

If we can find a solution to

$$\begin{cases} \partial_t z_1 = \Delta z_1 + v + \hat{u}_1 & \text{in } Q_T, \\ \partial_t z_2 = \Delta z_2 + \partial_{x_1}(z_1) + a z_2 + \hat{u}_2 & \text{in } Q_T, \\ \text{Supp}(z, v) \subset\subset \omega \times (0,T), \end{cases} \qquad \text{(S1)}$$

then $(\widehat{y} - z, -v)$ will be a solution to the initial problem.

Proof : first approach

Algebraic problem :

Let us write (S1) as

$$\mathcal{L}(z,v) = \widehat{u}$$

where

$$\mathcal{L}(z,v) = \begin{pmatrix} \partial_t z_1 - \Delta z_1 \\ \partial_t z_2 - \Delta z_2 - \partial_{x_1}(z_1) - a z_2 \end{pmatrix}.$$

Let us search $\mathcal{M}$ s.t.

$$\mathcal{L} \circ \mathcal{M} = Id \qquad \text{(S2)}$$

Thus $(z,v) := \mathcal{M}(\widehat{u})$ will be a solution to (S1).

Remark : $\widehat{u}$ have to be regular enough

Problem (S2) is equivalent to :

$$\mathcal{M}^* \circ \mathcal{L}^* \psi = \psi. \qquad \text{(S3)}$$

Resolution of the algebraic problem :

The operator $\mathcal{L}^*$ writes :

$$\mathcal{L}^*(\psi) = \begin{pmatrix} \mathcal{L}_1^*\psi \\ \mathcal{L}_2^*\psi \\ \mathcal{L}_3^*\psi \end{pmatrix} = \begin{pmatrix} -\partial_t\psi_1 - \Delta\psi_1 + \partial_{x_1}\psi_2 \\ -\partial_t\psi_2 - \Delta\psi_2 - a\psi_2 \\ \psi_1 \end{pmatrix}.$$

We remark that

$$\mathcal{L}_4^*\psi := \mathcal{L}_1^*\psi + (\partial_t + \Delta) \circ \mathcal{L}_3^*\psi = \partial_{x_1}\psi_2.$$

The commutator of $\mathcal{L}_1^*$ and $\mathcal{L}_4^*$ is :

$$[\mathcal{L}_1^* : \mathcal{L}_4^*]\psi = \partial_{x_1}(a)\psi_2.$$

So

$$\mathcal{M}_1^* \circ \mathcal{L}_1^* + \mathcal{M}_2^* \circ \mathcal{L}_2^* + \mathcal{M}_3^* \circ \mathcal{L}_3^* = Id.$$

Duality

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ \partial_t y_2 = \Delta y_2 + \partial_{x_1} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases}$$

Proposition

System (1) is **null controllable** at time T iff there exists $C_{obs} > 0$ s.t. for all $\varphi^0 \in L^2(\Omega)$ the solution to

$$\begin{cases} -\partial_t \varphi_1 = \Delta \varphi_1 - \partial_{x_1} \varphi_2 & \text{in } Q_T, \\ -\partial_t \varphi_2 = \Delta \varphi_2 + a\varphi_2 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot,T) = \varphi^0 & \text{in } \Omega \end{cases} \quad \text{(D)}$$

satisfies the **inequality of observability**

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C_{obs} \int_{q_T} \varphi_1^2.$$

Algebraic resolvability + Observability inequality :

Let φ be the solution to the dual system

$$\begin{cases} -\partial_t\varphi_1 = \Delta\varphi_1 - \partial_{x_1}\varphi_2 & \text{in } Q_T, \\ -\partial_t\varphi_2 = \Delta\varphi_2 + a\varphi_2 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot,T) = \varphi^0 & \text{in } \Omega. \end{cases}$$

Using the algebraic resolvability :

$$\varphi = \mathcal{M}_1^*(-\partial_t\varphi_1 - \Delta\varphi_1 + \partial_{x_1}\varphi_2) + \mathcal{M}_2^*(-\partial_t\varphi_2 - \Delta\varphi_2 - a\varphi_2) + \mathcal{M}_3^*\varphi_1 = \mathcal{M}_3^*\varphi_1$$

Then

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leqslant C\int_{q_T}\varphi^2 = C\int_{q_T}(\mathcal{M}_3^*\varphi_1)^2$$

Thus we have a solution to the control problem :

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathcal{M}_3 u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + \partial_{x_1} y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0,\ y(\cdot,T) = 0 & \text{in } \Omega. \end{cases}$$

Proof

Regular control using the fictitious control method :

Let $\widehat{\omega} \subset\subset \omega$ and $(\widehat{y}, \widehat{u})$, $\overline{y}$, θ, η s.t.

$$\begin{cases} \partial_t \widehat{y} = \Delta \widehat{y} + \mathbb{1}_{\hat{\omega}} \hat{u}, & \text{in } Q_T, \\ \widehat{y} = 0, & \text{on } \Sigma_T, \\ \widehat{y}(\cdot, 0) = y^0, \ \widehat{y}(\cdot, T) = 0, & \text{in } \Omega \end{cases} \qquad \begin{cases} \partial_t \overline{y} = \Delta \overline{y}, & \text{in } Q_T, \\ \overline{y} = 0, & \text{on } \Sigma_T, \\ \overline{y}(\cdot, 0) = y^0, & \text{in } \Omega \end{cases}$$

$$\begin{cases} \text{Supp}(\theta) \subset \omega, \\ \theta = 1 \text{ dans } \widehat{\omega}, \\ 0 \leq \theta \leq 1 \end{cases} \quad \text{and} \quad \begin{cases} \eta = 1 \text{ dans } [0, T/3], \\ \eta = 0 \text{ dans } [2T/3, T], \\ 0 \leq \eta \leq 1. \end{cases}$$

Then $y := (1 - \theta)\widehat{y} + \eta\theta\overline{y}$ is solution to

$$\begin{cases} \partial_t y = \Delta y + \mathbb{1}_{\omega} u & \text{in } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(\cdot, 0) = y^0, \ y(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

with $u := \Delta\theta\widehat{y} + 2\nabla\theta \cdot \nabla\widehat{y} + \partial_t(\eta)\theta\overline{y} - \eta\Delta\theta\overline{y} - 2\eta\nabla\theta \cdot \nabla\overline{y}$.
We remark that y is more regular than $\widehat{y}$.

Outline

1. **Presentation of the results**
2. **Algebraic resolvability**
3. **Proof of the results**
   - **Positive result**
   - **Negative result**
4. **Conclusion**

Negative result

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + \partial_x y_1 + a y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases}$$

Theorem (M. D., P. Lissy, submitted)

There exists $a \in \mathcal{C}^\infty([0,\pi])$ such that :

- There exists $(a,b) \subset\subset (0,\pi)$ s.t., for all $T > 0$, system (1) is **null controllable** (hence approximatively controllable) at time T.
- There exists $(a,b) \subset\subset (0,\pi)$ such that, for all $T > 0$, system (1) is **not approximatively controllable** (hence not null controllable) at time T.

Fattorini Criterion

Let $\Omega := (0,\pi)$, $\omega := (5\pi/12, 7\pi/12)$ and consider the system

$$\left\{\begin{array}{ll}\partial_t y_1 = \partial_{xx} y_1 + \mathbb{1}_\omega u & \text{in } Q_T,\\ \partial_t y_2 = \partial_{xx} y_2 + \partial_x y_1 + a y_2 & \text{in } Q_T,\\ y = 0 & \text{on } \Sigma_T,\\ y(0,\cdot) = y^0 & \text{in } \Omega,\end{array}\right. \quad (1)$$

where $u \in L^2(Q_T)$ and $a \in C^\infty(\overline{\Omega})$ will be later on specified.

Theorem

System (1) is **approximatively controllable** at time T iff for all $s \in \mathbb{C}$ and all $\varphi \in D(\Delta)$, we have

$$\left.\begin{array}{ll}-\partial_{xx}\varphi - \partial_x \psi = s\varphi & \text{in } \Omega\\ -\partial_{xx}\psi - a\psi = s\psi & \text{in } \Omega\\ \varphi = 0 & \text{in } \omega\end{array}\right\} \Rightarrow (\varphi,\psi) = (0,0).$$

Proof

Construction of ψ :

Consider ψ a function of $\mathcal{C}^\infty(\overline{\Omega}) \cap D(\Delta)$ satisfying

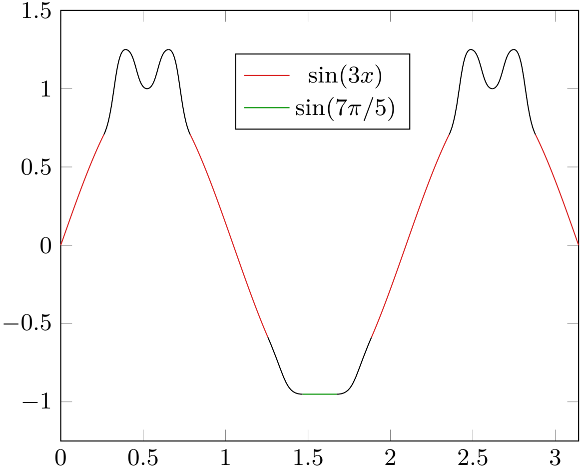
$$\begin{cases} \psi(x) = \sin(3x) + C_1\theta_1 + C_2\theta_2 + C_3\theta_3 + C_4\theta_4 \\ \qquad\qquad\qquad\qquad \text{for all } x \in [0, 4\pi/12] \cup [8\pi/12, \pi], \\ \psi(x) = \sin(7\pi/5) \text{ for all } x \in \overline{\omega}, \end{cases}$$

where $\theta_1,\ \theta_2,\ \theta_3,\ \theta_4$ are three nontrivial functions of $\mathcal{C}^\infty(\overline{\Omega})$ which will be chosen later on satisfying

$$\begin{cases} \mathrm{Supp}(\theta_1) \subset\subset (\pi/12, 2\pi/12), \\ \mathrm{Supp}(\theta_2) \subset\subset (2\pi/12, 3\pi/12), \\ \mathrm{Supp}(\theta_3) \subset\subset (8\pi/12, 9\pi/2), \\ \mathrm{Supp}(\theta_4) \subset\subset (9\pi/12, 10\pi/12), \\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4 \geqslant 0 \text{ in } \Omega, \end{cases}$$

$\varepsilon > 0$ small enough and $C_1,\ C_2,\ C_3,\ C_4$ are four positive constants to determined

Proof : Examples of function ψ



Proof

Construction of φ :

For a $\alpha \in \mathbb{R}$ to determined, the function $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ defined for all $x \in \overline{\Omega}$ by

$$\varphi(x) \quad := \quad \alpha \sin(3x) - \frac{1}{3}\int_0^x \sin(3(x-y))\partial_x\psi(y)dy$$

is solution to

$$-\partial_{xx}\varphi - \partial_x\psi = 9\varphi.$$

Since $\psi = \sin(7\pi/5)$ in ω,

$$\begin{aligned}\varphi(x) \quad = \quad & \left[\alpha - \frac{1}{3}\cos(7\pi/5)\sin(7\pi/5) - \int_0^{7\pi/15} \sin(3y)\psi(y)dy\right]\sin(3x) \\ & + \left[\frac{1}{3}\sin(7\pi/5)^2 - \int_0^{7\pi/15} \cos(3y)\psi(y)dy\right]\cos(3x),\end{aligned}$$

for all $x \in \omega$.

Proof

$\varphi = 0$ in ω :

Let us distinguish two cases :

1. If

$$\tfrac{1}{3}\sin(7\pi/5)^2 - \int_0^{6\pi/15} \cos(3y)\sin(3y)dy - \int_{6\pi/15}^{7\pi/15} \cos(3y)\psi(y)dy \quad (1)$$

is negative, since $\sin(3x), \cos(3x) > 0$ in $(2\pi/12, 3\pi/12)$, for $C_2 = 0$, one can chose C_1 such that

$$\tfrac{1}{3}\sin(7\pi/5)^2 - \int_0^{7\pi/15} \cos(3y)\psi(y)dy = 0. \qquad (2)$$

2. If the quantity (1) is positive, since $\sin(3x) > 0$ and $\cos(3x) < 0$ in $(3\pi/12, 4\pi/12)$, for $C_1 = 0$, one can chose $C_2 > 0$ such that (2) holds.

Thus, for α given by

$$\alpha := \frac{1}{3}\cos(7\pi/5)\sin(7\pi/5) + \int_0^{7\pi/15} \sin(3y)\psi(y)dy,$$

we obtain $\varphi = 0$ in ω.

Proof

$\varphi(\pi) = 0$:

By definition of φ, we have $\varphi(0) = 0$.
Now we search C_2, C_3such that $\varphi(\pi) = 0$.
We remark that

$$\varphi(\pi) = \frac{1}{3}\int_0^{\pi} \cos(3y)\psi(y)dy.$$

1. If

$$\tfrac{1}{3}\int_0^{2\pi/3} \cos(3y)\psi(y)dy + \frac{1}{3}\int_{2\pi/3}^{\pi} \cos(3y)\sin(3y)dy(2)$$

   is negative, then, using the fact that $\sin(3x),\ \cos(3x) > 0$ for all $x \in (9\pi/12, 5\pi/6)$, one can choose $C_3 := 0$ and find some some $C_2 > 0$ such that $\varphi(\pi) = 0$.

2. If now the quantity (2) is positive, since $\sin(3x) > 0$ and $\cos(3x) < 0$ for all $x \in (5\pi/6, 11\pi/12)$, one can choose $C_2 := 0$ and find some some $C_3 > 0$ such that $\varphi(\pi) = 0$.

Construction of a :

We define the function $a \in \mathcal{C}^\infty(\overline{\Omega})$ as follows

$$a := \frac{-\partial_{xx}\psi - 9\psi}{\psi} \in \mathcal{C}^\infty(\Omega).$$

Thus the three functions φ, ψ, $a \in \mathcal{C}^\infty(\overline{\Omega})$ satisfy

$$\begin{cases} -\partial_{xx}\varphi + \partial_x\psi = 9\varphi & \text{in } \Omega, \\ -\partial_{xx}\psi - a\psi = 9\psi & \text{in } \Omega, \\ \varphi(0) = \varphi(\pi) = \psi(0) = \psi(\pi) = 0, & \\ \varphi = 0 & \text{in } \omega, \\ (\varphi, \psi) \neq 0 & \text{in } \Omega. \end{cases}$$

Using Fattorini Criterion, System (S3) is not approximately controllable on the time interval $(0, T)$.

Outline

Setting

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \operatorname{div}(d_1 \nabla y_1) + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \operatorname{div}(d_2 \nabla y_2) + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (1)$$

where $y^0 \in L^2(\Omega; \mathbb{R}^2)$, $u \in L^2(Q_T)$, $g_{ij} \in L^\infty(Q_T; \mathbb{R}^N)$, $a_{ij} \in L^\infty(Q_T)$ for all $i, j \in \{1, 2\}$, $Q_T := \Omega \times (0, T)$, $\Sigma_T := (0, T) \times \partial\Omega$ and

$$\begin{cases} d_l^{ij} \in W^1_\infty(Q_T), \\ d_l^{ij} = d_l^{ji} \text{ in } Q_T, \end{cases} \qquad \sum_{i,j=1}^{N} d_l^{ij} \xi_i \xi_j \geqslant d_0 |\xi|^2 \text{ dans } Q_T, \ \forall \xi \in \mathbb{R}^N.$$

Boundary condition

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \mathrm{div}(d_1 \nabla y_1) + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \mathrm{div}(d_2 \nabla y_2) + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases} \tag{1}$$

Theorem (A. Benabdallah, M. Cristofol, P. Gaitan, L. De Teresa, 2014)

Suppose that $a_{ij} \in \mathcal{C}^4(\overline{Q}_T)$, $g_{ij} \in \mathcal{C}^3(\overline{Q}_T)^N$, $d_i \in \mathcal{C}^3(\overline{Q}_T)^{N^2}$ for all $i, j \in \{1, 2\}$ and

$$\begin{cases} \exists \gamma \neq \varnothing \text{ an open set of } \partial\omega \cap \partial\Omega, \\ \exists x_0 \in \gamma \text{ s.t. } g_{21}(t, x_0) \cdot \nu(x_0) \neq 0 \text{ for all } t \in [0, T], \end{cases}$$

where ν is the unitary exterior normal vector of $\partial\Omega$.
Then system (1) is **null controllable** at time T.

Constant case

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \mathrm{div}(d_1 \nabla y_1) + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \mathrm{div}(d_2 \nabla y_2) + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases} \tag{1}$$

Theorem (M. D., P. Lissy, 2016)

Suppose that d_i, g_{ij} et a_{ij} are constant in time and in space for all $i, j \in \{1, 2\}$.
Then system (1) is **null controllable** at time T if and only if

$$g_{21} \neq 0 \quad \text{or} \quad a_{21} \neq 0.$$

Let $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$\begin{cases} \partial_t y_1 = \operatorname{div}(d_1 \nabla y_1) + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \operatorname{div}(d_2 \nabla y_2) + \partial_{x_1} y_1 + g_{22} \cdot \nabla y_2 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases} \quad (1)$$

Theorem (M. D., P. Lissy, 2016, submitted)

Let $d_i^{kl}, g_{ij}^k \in \mathcal{C}^{N^2+3}(\overline{\omega}_T)$ and $a_{ij} \in \mathcal{C}^{N^2+2}(\overline{\omega}_T)$ for all $i, j \in \{1, 2\}$ and $k, l \in \{1, ..., N\}$. Suppose that $\exists\ \omega_T \subset q_T$ s.t.

$$\begin{cases} \widetilde{a}_{22} \text{ is not an element of the } \mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)\text{-module} \\ \left\langle 1, \widetilde{g}_{22}^2, ..., \widetilde{g}_{22}^N, d_2^{22}, ..., d_2^{NN} \right\rangle_{\mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)}, \end{cases}$$

where

$$\begin{cases} \widetilde{g}_{22}^i := g_{22}^i - \sum_{j=1}^N \partial_{x_j} d_2^{ij}, \\ \widetilde{a}_{22} := -a_{22} + \operatorname{div}(g_{22}). \end{cases}$$

Then system (1) is **null controllable** at time T.

Conclusion

- Positive results
  - Boundary condition
  - Coefficient a constant
  - Condition on a :

$$\partial_{x_1} a \neq 0 \text{ in } \omega$$

- Negative result
  - a well chosen
- Open problems
  - Minimal time of controllability
  - Is there a lot of coefficients a such that the system is not null controllable ?

Thank you for your attention !