On the determination of an initial heat distribution from a single measurement

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Summary

The material of this talk is taken from

M. Choulli, Various stability estimates for the problem of determining an initial heat distribution from a single measurement, arXiv :1512 :07421.

Inverse heat source problems appear in many branches of engineering and science. A typical application is for instance an accurate estimation of a pollutant source, which is a crucial environmental safeguard in cities with dense populations.

These inverse problems are severely ill-posed, involving a strongly time-irreversible parabolic dynamics. Their mathematical analysis is difficult and still a widely open subject.

Summary

- Approximation

  - Gilliam and Martin (Syst. & Control Letters 1987) considered the problem of recovering the initial data of the heat equation when the output is measured at points discrete in time and space... linked to the theory of Dirichlet series in 1d case.

  - Gilliam, Lund and Martin (Numer. Math. 1989) provide a simple and extremely accurate procedure for approximating the initial temperature for the heat equation on the line using a discrete time and spatial sampling...« sinc expansion ».

  - Li, Osher and Tsai (IPI 2014) considered the inverse problem of finding sparse initial data from the sparsely sampled solutions of the heat equation...ℓ^1 constrained optimization to find the initial data.

  - De Vore and Zuazua (M3AS 2014) studied the problem of approximating accurately the initial data in 1d case from finite measurements made at $(x_0, t_1), \ldots, (x_0, t_n)$... for a suitable choice of the point location of the sensor, x_0, $\exists\, t_1 < \ldots < t_n$ of time instants that guarantees the approximation with an « optimal rate ».

- Uniqueness

  - The determination of the initial distribution in the heat equation on a flat torus of arbitrary dimension was considered by Danger, Foote and Martin (Syst. Sci. Math. Sci. 1991) : observation of the solution along a geodesic determines uniquely the initial heat distribution if and only if the geodesic is dense in the torus...Fourier decomposition together with results from the theory of almost periodic functions.

  - Let u be the solution of the heat equation in the whole space $\mathbb{R}^d$. Nakamura, Saitoh and Syarif (IP 1999) showed that the initial distribution is determined and simply represented by the observations $u(t, x_0, x')$ and $\partial_{x_1} u(t, x_0, x')$, $t \geq 0$, $x' \in \mathbb{R}^{d-1}$, for some fixed $x_0 \in \mathbb{R}$.

- Stability

  Saitoh and al. obtained Lipschitz stability estimate from a point or a boundary observation...Reznitskaya transform and properties of Bergman-Selberg spaces.

Summary

Consider the IBVP for the one dimensional heat equation

$$\begin{cases} (\partial_t - \partial_x^2)u = 0 \text{ in } (0,\pi)\times(0,+\infty), \\ u(0,\cdot) = u(\pi,\cdot) = 0, \\ u(\cdot,0) = f. \end{cases} \quad (1)$$

Let $f \in L^2((0,\pi))$. Then $\exists!\, u_f \in C([0,+\infty), L^2((0,\pi)))$ solution of (1) :

$$u_f(x,t) = \frac{2}{\pi}\sum_{k\geq 1} \widehat{f}_k e^{-k^2 t}\sin(kx),$$

where $\widehat{f}_k$ is the k-th Fourier coefficient of f :

$$\widehat{f}_k = \frac{2}{\pi}\int_0^\pi f(x)\sin(kx)dx.$$

Inverse Problem : given a point $x_0 \in (0, \pi)$ for the placement of the sensor, we address the question of reconstructing the initial distribution f from $u_f(x_0, t)$, $t \in (0, T)$.

Set $a_k = \widehat{f}_k \sin(kx_0)$. The actual problem is reduced to one of recovering the sequence $a = (a_k)$ from the sum of the corresponding Dirichlet series

$$\sum_{k \geq 1} a_k e^{-k^2 t}.$$

x_0 has to be chosen in a strategic way so that $\sin(kx_0) \neq 0$ for all $k \geq 1$.

ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES

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ÉTUDE DES SOMMES D'EXPONENTIELLES RÉELLES

PAR

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PARIS
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1943

Summary

Notations :

The unit ball of a Banach space X is denoted by B_X.

$\ell^p = \ell^p(\mathbb{C})$, $1 \leq p < \infty$, is the usual Banach space of complex-valued sequences $a = (a_n)$ such that the series $\sum |a_n|^p$ is convergent :

$$\|a\|_{\ell^p} = \left(\sum_{k \geq 1} |a_k|^p \right)^{1/p}, \quad a = (a_k) \in \ell^p.$$

$\ell^\infty = \ell^\infty(\mathbb{C})$ denotes the usual Banach space of bounded complex-valued sequences $a = (a_n)$, normed by

$$\|a\|_\infty = \sup_k |a_k|, \quad a = (a_k) \in \ell^\infty.$$

For $\theta > 0$, the space $h^\theta = h^\theta(\mathbb{C})$ is defined as follows

$$h^\theta = \left\{ b = (b_k) \in \ell^2;\ \sum_{k \geq 1} \langle k \rangle^{2\theta} |b_k|^2 < \infty \right\},$$

where $\langle k \rangle = (1 + k^2)^{1/2}$.

h^θ is a Hilbert space when it is equipped with the norm

$$\|b\|_{h^\theta} = \left(\sum_{k \geq 1} \langle k \rangle^{2\theta} |b_k|^2 \right)^{1/2}.$$

Fix a sequence (λ_k) satisfying $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k < \ldots$ and $\lambda_k \to +\infty$ as k goes to $+\infty$.

To $a = (a_n) \in \ell^p$, $1 \leq p \leq \infty$, we associate the Dirichlet series

$$F_a(t) = \sum_{n \geq 1} a_n e^{-\lambda_n t}.$$

The series $F_a(t)$ converges for $t > 0$, for any $a \in \ell^p$, $1 \leq p \leq \infty$.

From the classical theory of Dirichlet series, F_a has an analytic extension to the half plane $\Re z > 0$. Since a Dirichlet series is zero if and only if its coefficients are identically equal to zero, we conclude that the knowledge of F_a in a subset of $\Re z > 0$ having an accumulation point determines uniquely a. In other words, if $D \subset \{\Re z > 0\}$ has an accumulation point and F_a vanishes on D, then $a = 0$.

The most interesting case is when $p = 1$. We can define in that case the operator $\mathcal{U}$ by

$$\begin{aligned} \mathcal{U} : \ell^1 &\longrightarrow C_b([0, +\infty)) \\ a &\longmapsto \mathcal{U}(a) := F_a, \; F_a(s) = \sum_{k \geq 1} a_k e^{-\lambda_k s}. \end{aligned}$$

Here, $C_b([0, +\infty))$ is the Banach space of bounded continuous function on $[0, +\infty)$, equipped with the supremum norm

$$\|F\|_\infty = \sup\{|F(s)|; \; s \in [0, +\infty)\}.$$

Then $\mathcal{U}$ is an injective linear contractive operator.

Similarly to entire series, we address the question to know whether it is possible to reconstruct the coefficients of a Dirichlet series from its sum. This is always possible if the values of the sum is known in the half plane $\Re z > 0$: for $\lambda_n < \lambda < \lambda_{n+1}$ and $\gamma > 0$,

$$\sum_{k=1}^{n} a_k = \frac{1}{2i\pi} \text{pv} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{F_a(z)}{z} e^{\lambda z} dz.$$

(D. V. Widder, An introduction to transform theory, Academic Press, New York, 1971.)

We aim to establish the modulus of continuity of the inverse of the mapping $a \in \ell^p \to F_{a|D}$, where D is a subset of $(0, +\infty)$:

$$\|a\|_{\ell^p} \leq \Psi\left(\|F_a\|_{L^\infty(D)}\right),$$

for a in some appropriate subset of ℓ^p, $p = 1$ or 2, where Ψ is a continuous, non decreasing and non negative real-valued function satisfying $\Psi(0) = 0$.

We discuss separately the cases $\Lambda := \sum_{k\geq 1} \frac{1}{\lambda_k} < \infty$ and $\Lambda = \infty$.

The nature of the series $\sum_{k\geq 1} \frac{1}{\lambda_k}$ is related to Müntz's theorem saying that the closure of the vector space spanned by $\{e^{-\lambda_k t};\ k \geq 1\}$ is dense in $C_0([0,+\infty)) = \{\varphi \in C_0([0,+\infty));\ \varphi(+\infty) = 0\}$ if and only if $\Lambda = \infty$. Usually this theorem is stated in the following equivalent form : $\{x^{\lambda_k};\ \ k \geq 1\}$ is dense in $C_0([0,1]) = \{\varphi \in C_0([0,1]);\ \varphi(0) = 0\}$ if and only if $\Lambda = \infty$.

When (λ_k) is the sequence of the Dirichlet eingenvalues, counted according to their multiplicity, of the laplacian on a bounded domain of $\mathbb{R}^d$, we know that $\lambda_k = O(k^{2/d})$. In that case $\Lambda < \infty$ holds if and only if $d = 1$.

In the sequel we use $F_a \in L^2((0,T))$ if $a \in \ell^2$.

Theorem 1

Assume that there exist $\beta > 1$, $K > 0$ *and* $\alpha > 0$ *so that*

$$\lambda_n = K(n+\alpha)^\beta + o(n^{\beta-1}), \quad n \to \infty. \tag{2}$$

Let B *a Lebesgue-measurable set of* $[0,T]$ *of positive Lebesgue measure. Let* $m > 0$ *and* $\theta > 0$, *there are two constants* $C > 0$ *and* $\delta > 0$ *so that, for any* $a \in mB_{h^\theta} \cap \ell^1$,

$$C\|a\|_{\ell^2} \leq \left|\ln\left(\frac{m}{\|F_a\|_{L^\infty(B)}}\right)\right|^{-\theta} + \|F_a\|_{L^\infty(B)}. \tag{3}$$

Observe that $h^\theta \subset \ell^1$ when $\theta > 1/2$. Therefore, $mB_{h^\theta} \cap \ell^1 = mB_{h^\theta}$ if $\theta > 1/2$.

Main ingredient in the proof : $\{e^{-\lambda_k t}\}$ has a biorthogonal set $\{\psi_k\}$ in $L^2(0,T)$:

$$\int_0^T \psi_k(t) e^{-\lambda_n t} dt = \delta_{nk},$$

so that

$$\|\psi_n\|_{L^2((0,T))} \leq Ce^{C\lambda_n^{1/\beta}}.$$

Remarks

1) We have the following Lipschitz stability estimate

$$\sum_{n\geq 1} e^{-cn}|a_n|^2 \leq C\|F_a\|_{L^2(0,T)}, \quad a \in \ell^2.$$

The left hand side of this inequality is seen as a ℓ^2-weighted norm of a.

2) One can replace (3) by, for an arbitrary $\theta > 0$,

$$C\|a\|_{\ell^2} \leq \left|\ln\left(\frac{m}{\|F_a\|_{L^2((0,T))}}\right)\right|^{-\theta} + \|F_a\|_{L^2((0,T))}, \quad a \in mB_{h^\theta}, \qquad (4)$$

3) It is possible to establish a Hölder stability estimate by replacing in Theorem 1 h^θ by the following subspace

$$h_{c,\gamma} = \left\{ b = (b_n);\ \sum_{n\geq 1} e^{cn^\gamma} |b_n|^2 < \infty \right\},$$

with $c > 0$ and $\gamma > 1$ appropriately chosen.

4) We can not in general replace in (4) the logarithmic stability by a Lipschitz or a Hölder stability. Indeed, if an estimate of the form, where $0 < \mu \leq 1$,

$$C\|a\|_{\ell^2} \leq \|F_a\|^{\mu}_{L^2((0,T))} + \|F_a\|_{L^2((0,T))}, \quad a \in B_{h^\theta}.$$

holds, then (by taking $\lambda_k = k^\beta$)

$$C \leq \frac{1}{k^{\mu\beta/2-\theta(1-\mu)}} + \frac{1}{k^{\beta/2}}, \quad k \geq 1.$$

But this inequality cannot be true if $\mu\beta/2 - \theta(1-\mu) > 0$.

Introduce the weighted ℓ^1 space, where $\theta > 0$,

$$\ell^{1,\theta} = \left\{ a = (a_i);\ \sum_{i\geq 1} i^\theta |a_i| < \infty \right\}.$$

This space is equipped with its natural norm

$$\|a\|_{\ell^{1,\theta}} = \sum_{i\geq 1} i^\theta |a_i|.$$

Assume that there are four constants $\beta_0 \geq 0$, $\beta_1 > 0$, $c > 0$ and $d > 0$ so that

$$\lambda_{i+1} - \lambda_i \geq \frac{d}{(i+1)^{\beta_0}} \text{ and } \lambda_i \leq c i^{\beta_1} \quad i \geq 1. \tag{5}$$

Theorem 2

Assume that (5) *holds and let* $m > 0$. *There are three constants* $c_0 > 0$, $c_1 > 0$ *and* δ, *that can depend only on* θ, β, c *and* d, *so that, for any* $a \in mB_{\ell^1,\theta}$,

$$c_0 \|a\|_{\ell^1} \leq |\ln(c_1|\ln(m\|F_a\|_\infty|))|^{-\theta/2} + \|F_a\|_\infty.$$

Even in the present case ($\Lambda = \infty$), it is possible to establish a Hölder stability estimate.

For $\alpha > 0$ and $\beta > 0$, consider the weighted ℓ^1-space

$$\ell^1_{\alpha,\beta} = \left\{ a = (a_i);\ \sum_{n\geq 1} e^{\alpha n^\beta} |a_i| < \infty \right\},$$

that we equip with its natural norm

$$\|u\|_{\ell^1_{\alpha,\beta}} = \sum_{n\geq 1} e^{\alpha n^\beta} |a_i|.$$

Theorem 3

Assume that (5) holds . Let $m > 0$, $\alpha > 0$ and $\beta > \beta_1$. There are there constants $C > 0$ and $\gamma > 0$, that can depend only on M, β_1, β, α and d, so that, for any $a \in mB_{\ell^1_{\alpha,\beta}}$,

$$C\|a\|_{\ell^1} \leq \|F_a\|_\infty^\gamma + \|F_a\|_\infty.$$

The main ingredient in the proof is a norm estimate of the inverse of a Vandermonde matrix :

W. Gautschi, Norm estimates for inverses of Vandermonde Matrices, Numer. Math. 23 (1975) 337-347.

Summary

We come back to the 1d heat equation. Consider the IBVP

$$\begin{cases} (\partial_t - \partial_x^2)u = 0 \ \text{ in } (0,\pi) \times (0,+\infty), \\ u(0,\cdot) = u(\pi,\cdot) = 0, \\ u(\cdot,0) = f. \end{cases} \tag{6}$$

The solution of the IBVP (6) is given by

$$u_f(x,t) = \frac{2}{\pi} \sum_{k\geq 1} \widehat{f}_k e^{-k^2 t} \sin(kx),$$

where $\widehat{f}_k$ is the Fourier coefficient of $f \in L^2((0,\pi))$:

$$\widehat{f}_k = \frac{2}{\pi} \int_0^\pi f(x) \sin(kx) dx.$$

Fix $x_0 \in (0,\pi)$ so that

$$|\sin(kx_0)| \geq d_0 k^{-1}, \quad k \geq 1,$$

where d_0 is an absolute constant

Recall that, where $\theta > 0$,

$$H^\theta((0,\pi)) = \{h \in L^2((0,\pi));\ \sum_{k\geq 1} \langle k\rangle^{2\theta} |\widehat{h}_k|^2 < \infty\}.$$

$H^\theta((0,\pi))$ is equipped with its natural norm

$$\|h\|_{H^\theta((0,\pi))} = \left(\sum_{k\geq 1} \langle k\rangle^{2\theta} |\widehat{h}_k|^2\right)^{1/2}.$$

A consequence of Theorem 1 is

Theorem 4

Let B a measurable set of $[0,T]$ of positive Lebesgue measure. Let $m>0$, there exists a constant $C>0$ so that, for any $f \in mB_{H^2((0,\pi))}$,

$$C\|f\|_{L^2((0,\pi))} \leq \left|\ln\left(\frac{2m/\pi}{\|u_f(x_0,\cdot)\|_{L^\infty(B)}}\right)\right|^{-1} + \|u_f(x_0,\cdot)\|_{L^\infty(B)}.$$

The previous result can be extended to a fractional 1d heat equation. For $\alpha > 0$, we define A^α, the fractional power of the operator $A = -\partial_x^2$ under Dirichlet boundary condition, as follows

$$A^\alpha f = \frac{2}{\pi} \sum_{k \geq 1} k^{2\alpha} \widehat{f}_k \sin(kx),$$

$$D(A^\alpha) = \{f \in L^2((0,\pi));\ \sum_{k \geq 1} k^{4\alpha} |\widehat{f}_k|^2 < \infty\}.$$

The IBVP for the heat equation for the fractional 1d Laplacian is represented by the Cauchy problem

$$\begin{cases} (\partial_t + A^\alpha) u = 0 \ \text{ in } (0, +\infty), \\ u(\cdot, 0) = f. \end{cases}$$

The solution of this Cauchy problem is given by, where $f \in L^2((0,\pi))$,

$$u_f^\alpha(x,t) = \frac{2}{\pi} \sum_{k \geq 1} e^{-k^{2\alpha} t} \widehat{f}_k \sin(kx).$$

If $1/2 < \alpha < 1$, we can apply again Theorem 1.

Theorem 5

Assume $1/2 < \alpha < 1$. Let B a Lebesgue-measurable set of $[0,T]$ of positive Lebesgue measure. Let $m > 0$, there exists a constant $C > 0$ so that, for any $f \in mB_{H^2((0,\pi))}$,

$$C\|f\|_{L^2((0,\pi))} \leq \left|\ln\left(\frac{2m/\pi}{\|u_f^\alpha(x_0,\cdot)\|_{L^\infty(B)}}\right)\right|^{-1} + \|u_f^\alpha(x_0,\cdot)\|_{L^\infty(B)}.$$

When $0 < \alpha \leq 1/2$, it follows from Theorem 2 :

Theorem 6

Let $m > 0$ and $\theta > 1/2$. There are three constants $c_0 > 0$ and $c_1 > 0$, that can depend only on θ, α and m, so that, for any $f \in mB_{H^{\theta+1}((0,\pi))}$,

$$\|f\|_{L^2((0,\pi))} \leq c_0 \left(\left| \ln(c_1 | \ln(mc_\theta \| u_f^\alpha(x_0,\cdot)\|_\infty)|) \right|^{-1/4} + \|u_f^\alpha(x_0,\cdot)\|_\infty \right).$$

for some constant c_θ.

Summary

Theorem 7

Let B a Lebesgue-measurable set of $[0,T]$ of positive Lebesgue measure and $\beta > 1/2$. Let $m > 0$, there exist two constants $C_0 > 0$ and $C_1 > 0$ so that, for any $f \in mB_{H^\beta((0,\pi))}$,

$$C_0\|f\|_{L^2((0,\pi))} \leq \left|\ln\left(\frac{C_1}{\|\partial_x u_f^\alpha(0,\cdot)\|_{L^\infty(B)}}\right)\right|^{-\frac{\beta}{\max(\alpha,\beta)}} + \|\partial_x u_f^\alpha(0,\cdot)\|_{L^\infty(B)}. \tag{7}$$

We have a variant of Theorem 7 in which (7) is replaced by

$$C_0\|f\|_{L^2((0,\pi))} \leq \left|\ln\left(\frac{C_1}{\|\partial_x u_f^\alpha(0,\cdot)\|_{L^2((0,T))}}\right)\right|^{-\frac{\beta}{\max(\alpha,\beta)}} + \|\partial_x u_f^\alpha(0,\cdot)\|_{L^2((0,T))},$$

without the restriction that $\beta > 1/2$. We have only to assume that $\beta > 0$.

Summary

Let $\Omega = \prod_{i=1}^{d}(0, \mu_i \pi)$, where the sequence $(\mu_1, \dots \mu_d)$ is non-resonant. In this case the Dirichlet-Laplacian on Ω has simple eigenvalues

$$\lambda_K = \prod_{i=1}^{d} \frac{k_i^2}{\mu_i^2}, \quad K = (k_1, \dots, k_d) \in \mathbb{N}^d, \; k_i \geq 1.$$

To each λ_K corresponds the eigenfunction

$$\varphi_K = \left(\frac{2}{\pi}\right)^d \frac{1}{\prod_{i=1}^{d} \mu_i} \prod_{i=1}^{d} \sin(k_i x_i / \mu_i)$$

so that (φ_K) forms an orthonormal basis of $L^2(\Omega)$.

Let $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. The fractional power A^α, $\alpha > 0$, is defined as follows

$$A^\alpha f = \sum_{K=(k_1,\ldots k_d)\in\mathbb{N}^d,\ k_i\geq 1} \lambda_K^\alpha (f, \varphi_K)\varphi_K.$$

$$D(A^\alpha) = \left\{ f \in L^2(\Omega); \sum_{K=(k_1,\ldots k_d)\in\mathbb{N}^d,\ k_i\geq 1} \lambda_K^{2\alpha} |(f, \varphi_K)|^2 < \infty \right\}.$$

Consider the Cauchy problem for the fractional heat equation associated to A^α :

$$\begin{cases} (\partial_t + A^\alpha)u = 0 \ \text{ in } (0, +\infty), \\ u(\cdot, 0) = f. \end{cases}$$

The solution of this Cauchy problem is given by

$$u_f^\alpha(x,t) = \sum_{K=(k_1,\ldots k_d)\in\mathbb{N}^d,\ k_i\geq 1} e^{-t\lambda_K^\alpha} (f, \varphi_K)\varphi_K.$$

Due to the difficulty of renumbering (λ_K) as a non decreasing sequence, we cannot proceed as in the 1d case. We restrict ourselves to the case $f = f_1 \otimes \ldots \otimes f_d \in \bigotimes_{i=1}^{d} C_0^\infty(0, \mu_i \pi)$. In that case,

$$u_f^\alpha(x,t) = \prod_{i=1}^{d} \sum_{k_i \geq 1} e^{-t k_i^{2\alpha}/\mu_i^{2\alpha}} (f_i, \varphi_{k_i}) \varphi_{k_i},$$

where

$$\varphi_{k_i} = \frac{2}{\mu_i \pi} \sin\left(\frac{k_i x_i}{\mu_i}\right).$$

In other words,

$$u_f^\alpha(x,t) = \prod_{i=1}^{d} u_{f_i}^\alpha(x_i, t).$$

Here $u_{f_i}^\alpha$ is the solution of 1d fractional heat equation in $(0, \mu_i \pi)$.

Set

$$\Gamma(f) := \max_{1\le j\le d} \|u_f^\alpha(\cdot,\ldots,\cdot,\mu_j x_0,\cdot,\ldots,\cdot)\|_{L^\infty(\prod_{i\neq j}(0,\mu_i\pi)\times(0,T))}.$$

The results for 1d case yield

$$\|f\|_{L^2(\Omega)} = \prod_{i=1}^{d} \|f_i\|_{L^2((0,\mu_i\pi))} \le C_0 \left(\left| \ln\left(\frac{C_1}{\Gamma(f)}\right)\right]^{-1} + \Gamma(f)\right).$$

By a continuity argument we can extend this estimate to f belonging to the closure of $\bigotimes_{i=1}^{d} C_0^\infty(0,\mu_i\pi)$ in $H^{2+(d-1)/2}(\Omega)$.