# Observability inequalities and inverse problems for evolution equations

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# Summary


1. Determining the initial heat distribution
2. Inverse source problem : an abstract framework
3. Inverse source problem for a damped WE
4. Determining the potential and the damping coefficient in a WE
5. Application to the clamped Euler-Bernoulli beam equation
6. Determining the potential in a WE without a geometric assumption
7. Extension to the heat equation
8. Determining a boundary coefficient in a wave equation

## Notations

Otherwise specified, $\Omega$ is $C^\infty$-smooth bounded domain of $\mathbb{R}^n$, $\tau > 0$,

$$Q = \Omega \times (0, \tau) \text{ and } \Sigma = \partial\Omega \times (0, \tau).$$

The normal exterior vector field to $\partial\Omega$ is denoted by $\nu = (\nu_1, \ldots \nu_n)$ and

$$\partial_\nu = \nu \cdot \nabla := \sum_{k=1}^{k} \nu_k \partial_k.$$

The norm of a normed space $X$ is denoted by $\| \cdot \|_X$ and the unit ball with respect to this norm is denoted by $B_X$.

If $X$, $Y$ are two Banach space, the space of bounded operators from $X$ into $Y$ (resp. $X$) is denoted by $\mathscr{B}(X, Y)$ (resp. $\mathscr{B}(X)$).

$H^{2,1}(Q)$ is the usual anisotropic Sobolev space :

$$H^{2,1}(Q) = L^2(0, \tau; H^2(\Omega)) \cap H^1(0, \tau; L^2(\Omega)).$$

## Preliminary comment

Most of the results of this course remain true if $\Omega$ is substituted by a **compact Riemannian manifold** with boundary $(M, g)$ :

$$g = g_{ij} dx^i \otimes dx^j$$

in the local coordinates system $(x_1, \ldots, x_n)$.

In that case instead of the Laplace operator we take the **Laplace-Beltrami** operator :

$$\Delta_g u = |g|^{-1/2} \partial_i \left( |g|^{1/2} g^{ij} \partial_j u \right),$$

where $(g^{ij})$ is the inverse of the metric $g$ and $|g|$ is the determinant of $g$.

## Definitions

- Let $X$ be a Banach space. A family $(T(t))_{t\geq 0}$ of $\mathscr{B}(X)$ is called a (strongly) **continuous semigroup** if

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$, $t, s \geq 0$ (semigroup property),

(iii) $\lim_{t\downarrow 0} T(t)x = x$, for any $x \in X$ (strong continuity).

A continuous semigroup of contractions $(T(t))$ is a continuous semigroup so that $\|T(t)\|_{\mathscr{B}(X)} \leq 1$ for any $t \geq 0$.

A (strongly) **continuous group** is a family $(T(t))_{t\in\mathbb{R}}$ of $\mathscr{B}(X)$ satisfying (i), (iii) and

(ii) $T(t+s) = T(t)T(s)$, $t, s \in \mathbb{R}$ (group property).

Define $A : X \to X$ by

$$D(A) = \{x \in X;\ \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\},$$
$$Ax := \lim_{t \downarrow 0} \frac{T(t)x - x}{t}, \quad x \in D(A).$$

This operator is called the (infinitesimal) **generator** of the semigroup $(T(t))$.

If $A$ generate a continuous semigroup then so is $A + B$, for any $B \in \mathscr{B}(X)$, with $D(A + B) = D(A)$.

- The **resolvent set** of an operator $A : D(A) \subset X \to X$ is given by

$$\rho(A) = \{\lambda \in \mathbb{C};\ (\lambda - A)^{-1} \in \mathscr{B}(X)\}.$$

The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the **spectrum** of $A$.

- Let $H=(H,(\cdot,\cdot),\|\cdot\|)$ be a (complex) Hilbert space. An operator $A: D(A)\subset H \rightarrow$ is called **dissipative** if

$$\Re(Ax,x)\leq 0,\;\; x\in D(A).$$

If in addition $R(\lambda - A)=H$ for some $\lambda\in\mathbb{C}$, then $A$ is called $m$-**dissipative**.

**Any $m$-dissipative operator is the generator of continuous semigroup of contractions. In particular, a $m$-dissipative operator has dense domain.**

Let $A: D(A)\subset H\rightarrow H$ densely defined and let $D^*$ be the set of $v\in H$ for which there exists $w\in H$ so that

$$(v,Au)=(w,u),\;\; u\in D(A).$$

The operator $A^*: D(A^*)=D^*\rightarrow H: v\rightarrow A^*v:=w$ is called the **adjoint** of $A$.

When $(A, D(A)) = (A^*, D(A^*))$ (resp. $(-A^*, D(A^*))$ we say that $A$ is **self-adjoint** (resp. **skew-adjoint**).

If $A$ is skew-adjoint, then $iA$ is self-adjoint.

A self-adjoint operator $A : D(A) \subset H \to H$ is called **positive** (resp. **strictly positive**) if $(Ax, x) \geq 0$ (resp. $(Ax, x) \geq \kappa \|x\|_H$ for some $\kappa > 0$) for any $x \in D(A)$.

**If $A$ is positive then $-A$ is $m$-dissipative.**

- Let $(e_k)$ be the standard basis of $\ell^2$. A sequence $(\phi_k)$ of $H$ is called a **Riesz basis** if there exists an invertible operator $Q \in \mathscr{B}(H, \ell^2)$ so that $Q\phi_k = e_k$ for each $k$.

Define $(\widetilde{\phi}_k)$ the **biorthogonal sequence** to $(\phi_k)$ by

$$\widetilde{\phi}_k = Q^* Q \phi_k.$$

Note that $(\widetilde{\phi}_k)$ is also a Riesz basis.

If $(\phi_k)$ is a Riesz basis of $H$ and $(\widetilde{\phi}_k)$ the biorthogonal sequence to $(\phi_k)$, then

$$x = \sum_k (x, \widetilde{\phi}_k)\phi_k, \quad x \in H.$$

Moreover, there exist $0 < \alpha < \beta$ so that, for any $x \in H$, we have

$$\alpha\|x\| \leq \left(\sum_k \left|(x, \widetilde{\phi}_k)\right|^2\right)^{1/2} \leq \beta\|x\|.$$

- $A \in \mathscr{B}(H)$ is **compact** if $AB_H$ is relatively compact.

**Assume that $H$ is separable. Let $A$ be a self-adjoint compact operator. Then there exists a sequence $(\phi_k)$ of eigenvectors forming an orthonormal basis of $H$. In the particular case when $N(A) = \{0\}$, $\sigma(A) = (\mu_k)$ with $\mu_k \to 0$,**

$$x = \sum_k (x, \phi_k)\phi_k, \quad Ax = \sum_k \mu_k (x, \phi_k)\phi_k, \quad x \in H.$$

## Laplace operator with Dirichlet boundary condition

we endow $H_0^1(\Omega)$ with the norm

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)^n}, \quad u \in H_0^1(\Omega). \tag{1}$$

Note that in light of **Poincarré's inequality**

$$\|u\|_{L^2(\Omega)} \leq c_\Omega \|\nabla u\|_{L^2(\Omega)^n}, \quad u \in H_0^1(\Omega).$$

On $L^2(\Omega)$, consider

$$A = \Delta = \sum_{i=1}^{n} \partial_i^2, \quad D(A) = \{u \in H_0^1(\Omega);\ \Delta u \in L^2(\Omega)\}.$$

According to the **elliptic regularity** $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

From **Green's formula**

$$\int_\Omega \Delta u \overline{v} dx = -\int_\Omega \nabla u \cdot \overline{\nabla v} dx = \int_\Omega u \overline{\Delta v}, \quad u, v \in D(A).$$

Therefore $A$ is self-adjoint and

$$\int_\Omega \Delta u \overline{u} dx = -\int_\Omega |\nabla u|^2 dx \leq 0, \quad u \in D(A).$$

Whence, $A$ is dissipative.

Let $f \in L^2(\Omega)$. Since

$$a(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} dx$$

is a scalar product associated to the norm $\| \cdot \|_{H_0^1(\Omega)}$, by **Riesz representation theorem**, there exists a unique $u_f \in H_0^1(\Omega)$ so that

$$a(u_f, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} dx = \int_\Omega f \overline{v} dx, \quad v \in H_0^1(\Omega).$$

This identity entails $-\Delta u_f = f$ in $\mathscr{D}'(\Omega)$. Hence $u_f \in D(A)$ and by the usual elliptic a priori estimate,

$$\|u_f\|_{H^2(\Omega)} \leq c_\Omega \|f\|_{L^2(\Omega)}.$$

That is $A : D(A) \to L^2(\Omega)$, where $D(A)$ is equipped with the graph norm, has a bounded inverse $A^{-1} : L^2(\Omega) \to D(A)$. Consequently, $A^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact : $\sigma(-A) = (\lambda_k)$, a sequence of positive real number satisfying $\lambda_k \to +\infty$, and $L^2(\Omega)$ has an **orthonormal basis** $(\phi_k)$ consisting of **eigenfunctions**.

In that case

$$u = \sum_k (u, \phi_k)\phi_k, \quad u \in L^2(\Omega),$$

$$Au = \sum_k \lambda_k (u, \phi_k)\phi_k, \quad u \in D(A),$$

$$e^{tA}u = \sum_k e^{-\lambda_k t}(u, \phi_k)\phi_k, \quad u \in L^2(\Omega),\ t \geq 0.$$

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Consider on $L^2(\Omega)$ the unbounded operator defined by

$$A = -\Delta \;\text{ and }\; D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_k \leq \ldots$ the sequence of eigenvalues of $A$, counted according to their multiplicity, and let $(\phi_k)$ the corresponding sequence of eigenfunctions forming an orthonormal basis of $L^2(\Omega)$.

For any $f \in H_0^1(\Omega)$, the IBVP for the heat equation

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = f. & \end{cases} \tag{2}$$

has a unique solution $u_f \in H^{2,1}(Q)$ with $\partial_\nu u_f \in L^2(\Sigma)$.

Let $\Gamma$ be a non empty open subset of $\partial\Omega$ and $\omega$ be a non empty open subset of $\Omega$. Set $\Sigma_\Gamma = \Gamma \times (0,\tau)$ and $Q_\omega = \omega \times (0,\tau)$.

The following two **final time observability inequalities** hold

$$\|u_f(\cdot,\tau)\|_{H_0^1(\Omega)} \leq C\|\partial_\nu u_f\|_{L^2(\Sigma_\Gamma)}, \quad f \in H_0^1(\Omega), \tag{3}$$

and

$$\|u_f(\cdot,\tau)\|_{L^2(\Omega)} \leq C\|u_f\|_{L^2(Q_\omega)}, \quad f \in L^2(\Omega). \tag{4}$$

Here $u_f$ is the solution of the IBVP (2) and $C$ is a constant independent on $f$.

Let $H_0^s(\Omega) = \{w \in H^s(\Omega);\ u = 0 \text{ on } \partial\Omega\}$, $s > 1/2$. Then

$$H_0^{2\theta}(\Omega) = \left\{ w \in L^2(\Omega);\ \sum_{k\geq 1} \lambda_k^{2\theta} |(w,\phi_k)|^2 < \infty \right\} \text{ if } 1/4 < \theta < 3/4.$$

**Theorem 1**

*Let $1/2 \le \theta < 3/4$ and $m > 0$. Then there exists a constant $C > 0$, depending on $\Omega$, $\Gamma$ (resp. $\omega$), $\theta$ and $m$, so that, for any $f \in mB_{H_0^{2\theta}(\Omega)}$,*

$$C\|f\|_{L^2(\Omega)} \le \left|\ln \|\partial_\nu u_f\|_{L^2(\Sigma_\Gamma)}\right|^{-\theta} + \|\partial_\nu u_f\|_{L^2(\Sigma_\Gamma)} \tag{5}$$

*and*

$$C\|f\|_{L^2(\Omega)} \le \left|\ln \|u_f\|_{L^2(Q_\omega)}\right|^{-\theta} + \|u_f\|_{L^2(Q_\omega)}. \tag{6}$$

**Sketch of the proof**

From

$$u_f(\cdot, t) = \sum_{k\ge 1} e^{-\lambda_k t}(f, \phi_k)\phi_k,$$

we get

$$(f, \phi_k) = e^{\lambda_k \tau}(u_f(\cdot, \tau), \phi_k). \tag{7}$$

For $\lambda \geq \lambda_1$, denote by $N = N(\lambda)$ the integer satisfying $\lambda_N \leq \lambda < \lambda_{N+1}$. Therefore

$$\begin{aligned}
\|f\|_{L^2(\Omega)}^2 &= \sum_{k\geq 1} (f, \phi_k)^2 \\
&= \sum_{k\leq N} (f, \phi_k)^2 + \sum_{k>N} (f, \phi_k)^2 \\
&\leq \sum_{k\leq N} (f, \phi_k)^2 + \frac{1}{\lambda^{2\theta}} \sum_{k>N} \lambda_k^{2\theta} (f, \phi_k)^2 \\
&\leq \sum_{k\leq N} (f, \phi_k)^2 + \frac{m^2}{\lambda^{2\theta}}.
\end{aligned}$$

This and (7) entail

$$\|f\|_{L^2(\Omega)}^2 \leq N(\lambda) e^{2\lambda\tau} \|u_f(\cdot, \tau)\|_{L^2(\Omega)}^2 + \frac{m^2}{\lambda^{2\theta}}. \tag{8}$$

Combined with (3), (8) yields

$$\|f\|^2_{L^2(\Omega)} \leq e^{c\lambda}\|\partial_\nu u_f\|_{L^2(\Sigma_\Gamma)} + \frac{m^2}{\lambda^{2\theta}},$$

where we used that $N(\lambda) \leq C\lambda^{n/2}$.

A standard **minimization** argument, with respect to $\lambda$, gives then (5).

The proof of (6) is quite similar. We have only to use the observability inequality (4) instead of (3).

## Comments

• We can substitute in (6), $\|u_f\|_{L^2(Q_\omega)}$ by $\|u_f\|_{L^2(D)}$, where $D$ is any **Lebesgue-measurable** set contained in $\Omega \times (0, \tau)$, having a non zero Lebesgue measure. Also, if $\partial\Omega$ contains a **real-analytic** open sub-manifold $\Gamma_a$, then (5) still holds whenever $\Sigma_\Gamma$ is substituted by any Lebesgue-measurable subset of $\Gamma_a \times (0, \tau)$ with non zero Lebesgue measure.

- From a result by **Fernàndez-Cara** and **Zuazua**, there exist two constants $C_1 > 0$ and $C_2 > 0$, depending only on $\Omega$, $\omega$ and $\tau$, so that

$$\sum_{k\geq 1} e^{-C_1\sqrt{\lambda_k}}(f,\phi_k)^2 \leq C_2 \int_{Q_\omega} u_f^2 dxdt, \quad f \in L^2(\Omega),$$

which is equivalent to

$$\sum_{k\geq 1} e^{-C_1 k^{1/n}}(f,\phi_k)^2 \leq C_2 \int_{Q_\omega} u_f^2 dxdt, \quad f \in L^2(\Omega). \tag{9}$$

As the mapping $f \to \|f\|_{L^2_w(\Omega)} = \left(\sum_{n\geq 1} e^{-C_1 n^{1/d}}(f,\phi_k)^2\right)^{1/2}$ defines a norm on $L^2(\Omega)$, (9) can be reinterpreted as a **Lipschitz stability** estimate of determining $f$ from $u_{f\,|Q_\omega}$ :

$$\|f\|_{L^2_w(\Omega)} \leq C_2 \|u_f\|_{L^2(Q_\omega)}, \quad f \in L^2(\Omega).$$

Also, estimate (9) entails

$$(f,\phi_k)^2 \leq C_2 e^{C_1 k^{1/n}} \int_{Q_\omega} |u_f|^2 dxdt, \quad f \in L^2(\Omega), \quad k \geq 1.$$

This estimate allows us to retrieve the estimate (6).

- In the case of an internal measurement, we can establish a Hölder stability estimate by using a **Lebeau-Robbiano** type inequality for the eigenfunctions $\phi_n$. For $f \in L^2(\Omega)$, set

$$u_N(\cdot,t) = \sum_{k=1}^{N} e^{-\lambda_k t}(f,\phi_k)\phi_k.$$

If $\omega$ is a Lebesgue-measurable subset of $\Omega$ of positive Lebesgue measure, we have

$$\sum_{k=1}^{N} e^{-2\lambda_k t}(f,\Phi_k)^2 \leq Ce^{C\sqrt{\lambda_N}} \int_\omega u_N(\cdot,t)^2 dx.$$

But

$$\int_\omega u_N(\cdot,t)^2 dx \leq \int_\omega u_f(\cdot,t)^2 dx + \sum_{k\geq N+1} (f,\phi_k)^2.$$

Hence

$$\sum_{k=1}^{N} e^{-2\lambda_k t}(f,\phi_k)^2 \le Ce^{C\sqrt{\lambda_N}}\left[\int_\omega u_f(\cdot,t)^2 dx + \sum_{k\ge N+1}(f,\phi_k)^2\right].$$

Whence

$$\sum_{k=1}^{N}(f,\phi_k)^2 \le Ce^{C\lambda_N}\left[\int_{Q_\omega} u_f^2 dx + \sum_{k\ge N+1}(f,\phi_k)^2\right],$$

implying

$$\begin{aligned}\sum_{k\ge 1}(f,\phi_k)^2 &\le Ce^{C\lambda_N}\left[\int_{Q_\omega} u_f^2 dx + \sum_{k\ge N+1}(f,\phi_k)^2\right]\\ &\le Ce^{CN^{2/n}}\left[\int_{Q_\omega} u_f^2 dx + \sum_{k\ge N+1}(f,\phi_k)^2\right].\end{aligned}$$

Therefore, under the assumption

$$\sum_{k\geq 1} e^{ck^\gamma}(f,\Phi_k)^2 \leq m,$$

for some $c>0$, $m>0$ and $\gamma>2/n$, one obtains similarly to Theorem 1 the following **Hölder stability** estimate

$$\|f\|_{L^2(\Omega)} \leq C\left(\|u_f\|_{Q_\omega}^\theta + \|u_f\|_{Q_\omega}\right).$$

## Proof of the final time observability inequality

Let $\Gamma$ be a nonempty open subset of $\partial\Omega$. Then there exists $\psi \in C^4(\mathbb{R}^n)$ satisfying

(i) $\psi > 0$ in $\overline{\Omega}$,

(ii) $|\nabla\psi] \geq c$ in $\overline{\Omega}$, for some $c > 0$,

(iii) $\partial_\nu \psi \leq 0$ on $\partial\Omega \setminus \Gamma$.

Let $g(t) = \frac{1}{t(\tau - t)}$, $\Sigma_\Gamma = \Gamma \times (0, \tau)$ and set

$$\varphi = \varphi(x,t) = g(t)\left(e^{\rho\psi(x)} - e^{-2\rho\|\psi\|_{L^\infty(\Omega)}}\right).$$

**Carleman inequality** : there exist $C > 0$, $\rho > 0$ and $\lambda_0 > 0$ so that, for any $u \in H^{2,1}(Q)$ satisfying $u = 0$ on $\Sigma$ and $\lambda \geq \lambda_0$, we have

$$C \int_Q e^{2\lambda\varphi} \left[ (\lambda g)|\nabla u|^2 + (\lambda g)^3 |u|^2 \right] dxdt \qquad (10)$$

$$\leq \int_Q e^{2\lambda\varphi} |(\partial_t - \Delta)u|^2 dxdt + \int_{\Sigma_\Gamma} e^{2\lambda\varphi} (\lambda g) |\partial_\nu u|^2 ds(x)dt.$$

Recall that, for any $(f, F) \in H_0^1(\Omega) \oplus L^2(\Omega)$, the IBVP

$$\begin{cases} (\partial_t - \Delta)u = F & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = f. \end{cases}$$

has a unique solution $u = u(f, F) \in H^{2,1}(Q) \cap C([0, T]; H_0^1(\Omega))$ so that

$$\|u(\cdot, t)\|_{H_0^1(\Omega)} \leq C\|(f, F)\|_{H_0^1(\Omega) \oplus L^2(\Omega)}, \quad t \in [0, \tau]. \qquad (11)$$

Here the constant $C$ is independent on $(f, F)$.

Set $u_f = u(f,0)$, $f \in H_0^1(\Omega)$.

**Observability inequality** : there exists $C > 0$ so that, for any $f \in H_0^1(\Omega)$, we have

$$\|u_f(\cdot,\tau)\|_{H_0^1(\Omega)} \leq C\|\partial_\nu u_f\|_{L^2(\Sigma_\Gamma)}. \tag{12}$$

To prove this inequality, we first note that a straightforward application of (10) yields

$$\|u_f\|_{L^2(\Omega\times(\frac{\tau}{4},\frac{3\tau}{4}))} \leq C\|\partial_\nu u_f\|_{L^2(\Sigma_\Gamma)}. \tag{13}$$

Let $\theta \in C^\infty[0,\tau]$ so that $0 \leq \theta \leq 1$, $\psi = 0$ on $\left[0,\frac{\tau}{4}\right]$ and $\theta = 1$ on $\left[\frac{3\tau}{4},\tau\right]$. Since $v_f := \theta u = u(0,\theta' u_f)$, estimate (11) entails

$$\|v_f(\cdot,\tau)\|_{H_0^1(\Omega)} \leq C\|\theta' u_f\|_{L^2(Q)}$$

Whence

$$\|u_f(\cdot,\tau)\|_{H_0^1(\Omega)} = \|v_f(\cdot,\tau)\|_{H_0^1(\Omega)} \leq C\|\theta'\|_{L^\infty(0,\tau)}\|u_f\|_{L^2(\Omega\times(\frac{\tau}{4},\frac{3\tau}{4}))}. \tag{14}$$

Estimate (13) in (14) gives (12).

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Let $H$ be a Hilbert space and $A : D(A) \subset H \to H$ be the generator of continuous semigroup $T(t)$.

Let $Y$ be another Hilbert space that we identify with its dual space.

An operator $C \in \mathscr{B}(D(A), Y)$ is called an **admissible observation** for $T(t)$ if for some (and hence for all) $\tau > 0$, the operator $\Psi \in \mathscr{B}(D(A), L^2(0, \tau; Y))$ given by

$$(\Psi x)(t) = CT(t)x, \quad t \in [0, \tau], \quad x \in D(A),$$

has a bounded extension to $H$.

Consider the system

$$z'(t) = Az(t), \quad z(0) = x, \tag{15}$$
$$y(t) = Cz(t). \tag{16}$$

When $C$ is an admissible observation for $T(t)$, we say that the pair $(A, C)$ is **exactly observable** at time $\tau > 0$ if there is a constant $\kappa$ such that the solution $(z, y)$ of (15) and (16) satisfies

$$\int_0^\tau \|y(t)\|_Y^2 dt \geq \kappa^2 \|x\|_H^2, \quad x \in D(A).$$

Or equivalently

$$\int_0^\tau \|(\Psi x)(t)\|_Y^2 dt \geq \kappa^2 \|x\|_H^2, \quad x \in D(A). \tag{17}$$

Consider the **Cauchy problem**

$$z'(t) = Az(t) + \lambda(t)x, \quad z(0) = 0, \tag{18}$$

and set

$$y(t) = Cz(t), \quad t \in [0, \tau]. \tag{19}$$

**Duhamel's formula** yields

$$y(t) = \int_0^t \lambda(t-s)CT(s)xds = \int_0^t \lambda(t-s)(\Psi x)(s)ds. \tag{20}$$

Let $H^1_\ell(0, \tau; Y) = \{u \in H^1(0, \tau; Y); \ u(0) = 0\}$ and define the operator

$$S : L^2(0, \tau; Y) \longrightarrow H^1_\ell(0, \tau; Y)$$

by

$$(Sh)(t) = \int_0^t \lambda(t-s)h(s)ds. \tag{21}$$

If $E = S\Psi$, (20) takes the form

$$y(t) = (Ex)(t).$$

## Theorem 2

*Assume that* $(A, C)$ *is exactly observable for* $\tau > 0$*. Let* $\lambda \in H^1(0, \tau)$ *satisfies* $\lambda(0) \neq 0$*. Then* $E$ *is one-to-one from* $H$ *onto* $H^1_\ell(0, \tau; Y)$ *and*

$$\frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau \frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|x\|_H \leq \|Ex\|_{H^1_\ell(0,\tau;Y)}, \quad x \in H. \tag{22}$$

**Sketch of the proof**

Take the derivative with respect to $t$ of each side of the integral equation

$$\int_0^t \lambda(t-s)\varphi(s)ds = \psi(t),$$

to get the **Volterra integral equation of second kind**

$$\lambda(0)\varphi(t) + \int_0^t \lambda'(t-s)\varphi(s)ds = \psi'(t).$$

It is known that this integral equation has a unique solution $\varphi \in L^2(0,\tau;Y)$.

**Cauchy-Schwarz's** inequality together with an elementary convexity inequality give

$$|\lambda(0)|^2\|\varphi(t)\|_Y^2 \leq 2\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}\int_0^t |\varphi(0)|^2\|\varphi(s)\|_Y^2 ds + 2\|\psi'(t)\|_Y^2$$

An application of **Gronwall's lemma** yields

$$\|\varphi\|_{L^2(0,\tau;Y)} \leq \frac{\sqrt{2}}{|\lambda(0)|} e^{\tau\frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|\psi'\|_{L^2(0,\tau;Y)}$$

and then

$$\|\varphi\|_{L^2(0,\tau;Y)} \leq \frac{\sqrt{2}}{|\lambda(0)|} e^{\tau\frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|S\varphi\|_{H^1_\ell(0,\tau;Y)}.$$

In light of (17), one gets

$$\|Ex\|_{H^1_\ell(0,\tau;Y)} \geq \frac{\kappa|\lambda(0)|}{\sqrt{2}} e^{-\tau\frac{\|\lambda'\|^2_{L^2((0,\tau))}}{|\lambda(0)|^2}} \|x\|_H.$$

We get a variant of Theorem 2 by using the following fact : if $(A, C)$ is exactly observable for $\tau$ with constant $\kappa$, then there is $\delta > 0$ so that, for any $P \in \mathscr{B}(H)$ satisfying $\|P\| \leq \delta$, $(A + P, C)$ is exactly observable for $\tau$ with constant $\geq \kappa/2$.

Define $E^P$ similarly to $E$ by replacing $A$ by $A + P$.

**Theorem 3**

*We assume that $(A, C)$ is exactly observable for $\tau > 0$. Let $\lambda \in H^1(0, T)$ satisfies $\lambda(0) \neq 0$. There is $\delta > 0$ such that, for any $P \in \mathscr{B}(H)$ satisfying $\|P\| \leq \delta$, $E^P$ is one-to-one from $H$ onto $H^1_\ell(0, \tau; Y)$ and*

$$\frac{\kappa|\lambda(0)|}{2\sqrt{2}} e^{-\tau \frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|x\|_H \leq \|E^P x\|_{H^1_\ell(0,\tau;Y)}, \quad x \in H. \tag{23}$$

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Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma, \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1, \end{cases} \tag{24}$$

For $q, a \in L^\infty(\Omega,\mathbb{R})$ and $(u_0,u_1) \in \mathcal{H} := H_0^1(\Omega) \oplus L^2(\Omega)$, the IBVP (24) has a unique solution $u_{q,a} \in C([0,\tau], H_0^1(\Omega))$ so that $\partial_t u_{q,a} \in C([0,\tau], L^2(\Omega))$. Moreover

$$\|u_{q,a}\|_{C([0,\tau],H_0^1(\Omega))} + \|\partial_t u_{q,a}\|_{C([0,\tau],L^2(\Omega))} \leq C\|(u_0,u_1)\|_{\mathcal{H}}.$$

Note that the constant $C$ above is a non decreasing function of $\|q\|_\infty + \|a\|_\infty$.

Define on $\mathcal{H}$

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \oplus H_0^1(\Omega) := \mathcal{H}_1$$

and $\mathcal{A}_{q,a} = \mathcal{A}_0 + \mathcal{B}_{q,a}$ with $D(\mathcal{A}_{q,a}) = D(\mathcal{A}_0)$, where

$$\mathcal{B}_{q,a} = \begin{pmatrix} 0 & 0 \\ -q & -a \end{pmatrix}.$$

Let

$$\mathcal{C} : D(\mathcal{A}_0) \to L^2(\Sigma) : \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \longrightarrow \partial_\nu \varphi.$$

Let us give sufficient conditions ensuring that the pair $(\mathcal{A}_{q,a},\mathcal{C})$ is exactly observable. Fix $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and set

$$\Gamma_0 = \{x \in \partial\Omega;\ \nu(x)\cdot(x-x_0) > 0\} \ \text{ and } \ d = \max_{x\in\overline{\Omega}} |x-x_0|.$$

Let us assume that $\Gamma \supset \Gamma_0$. A classical result shows that $(\mathcal{A}_0,\mathcal{C})$ is exactly observable with $\tau \geq \tau_0 = 2d$. Then a perturbation argument enables to prove that $(\mathcal{A}_{q,a},\mathcal{C})$ is also exactly observable for $\tau \geq \tau_0$, again with $\tau_0 = 2d$.

Sharp sufficient conditions on $\Gamma$ and $\tau_0$ were established by **Bardos-Lebeau-Rauch**.

We are going to apply Theorem 3 to the **inverse source problem** associated with the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u + a(x)\partial_t u = \lambda(t)f(x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = 0. \end{cases} \tag{25}$$

Fix $(q_0, a_0) \in L^\infty(\Omega) \oplus L^\infty(\Omega)$ and assume that $(\mathcal{A}_{q_0,a_0}, \mathcal{C})$ is exactly observable with constant $\kappa$. Let $\beta > 0$ be so that, for any $(q,a) \in \mathcal{D} = (q_0, a_0) + \beta B_{W^{1,\infty}(\Omega)} \times B_{L^\infty(\Omega)}$, $(\mathcal{A}_{q,a}, \mathcal{C})$ is exactly observable with constant $\kappa/2$.

When $q_0 \geq 0$ we use the notation $\mathcal{D}^+$ instead of $\mathcal{D}$.

## Corollary 4

*If $u_f$ is the solution of the IBVP (25) with $(q,a) \in \mathcal{D}$ and $f \in L^2(\Omega)$, then*

$$\|f\|_2 \leq \frac{2\sqrt{2}}{\kappa|\lambda(0)|} e^{\tau \frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}} \|\partial_\nu u_f\|_{H^1(0,\tau;L^2(\Gamma))}.$$

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If $u_{q,a} = u_{q,a}(u_0, u_1)$ is the solution of the IBVP (24), then it is known that $\partial_\nu u_{q,a} \in L^2(\Sigma)$ and

$$\|\partial_\nu u_{q,a}(u_0, u_1)\|_{L^2(\Sigma)} \le C\|(u_0, u_1)\|_{\mathcal{H}}. \tag{26}$$

where $C$ is a non decreasing function of $\|q\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega)}$.

Define the **initial-to-boundary operator** $\Lambda_{q,a}$ by

$$\Lambda_{q,a} : (u_0, u_1) \in \mathcal{H} \mapsto \partial_\nu u_{q,a}(u_0, u_1) \in L^2(\Sigma).$$

Using $\partial_t u_{q,a}(u_0, u_1) = u_{q,a}(u_1, \Delta u_0 - qu_0 - au_1)$, one gets that

$$\Lambda_{q,a} \in \mathscr{B}(\mathcal{H}_1, H^1(0, \tau; L^2(\Gamma)).$$

Additionally

$$\|\Lambda_{q,a}\|_{\mathscr{B}(\mathcal{H}_1, H^1(0,\tau;L^2(\Gamma)))} \le C.$$

Here $C$ is as in (26).

For $m>0$, set $\mathcal{D}_m=\{(q,a)\in \mathcal{D}^+\cap H^2(\Omega)\oplus H^2(\Omega);\ \|(q,a)\|_{H^2(\Omega)\oplus H^2(\Omega)}\leq m\}$. Let $m_0$ be sufficiently large so that $\mathcal{D}_m\neq\emptyset$, for all $m\geq m_0$. Henceforth $m\geq m_0$ is fixed.

We first establish a stability estimate around zero **damping coefficient**.

**Theorem 5**

*Let $(q,0)\in\mathcal{D}_m$ with $q\in C^1(\overline{\Omega})$ and $q\geq 0$. Then there exists a constant $C>0$, that can depend on the data and $q$, so that, for any $(\widetilde{q},\widetilde{a})\in\mathcal{D}_m$, we have*

$$\|\widetilde{q}-q\|_{L^2(\Omega)}+\|\widetilde{a}-0\|_{L^2(\Omega)}\leq C\|\Lambda_{\widetilde{q},\widetilde{a}}-\Lambda_{q,0}\|^{1/2}.$$

*Here $\|\cdot\|$ is the norm of $\mathscr{B}(\mathcal{H}_1,H^1(0,\tau;L^2(\Gamma)))$.*

The proof of Theorem 5 is based on the following **weighted interpolation inequality** :

**Proposition 1**

*Let $q \in C^1(\overline{\Omega})$, $q \leq 0$, and $u \in C^2(\Omega) \cap H_0^1(\Omega)$ non identically equal to zero satisfying $\Delta u + qu \leq 0$. There exists a constant $c_u$, that can depend only on $u$ and $\Omega$ so that, for any $f \in H^2(\Omega)$,*

$$\|f\|_{L^2(\Omega)} \leq c_u \|fu\|_{L^2(\Omega)}^{1/2} \|f\|_{H^2(\Omega)}^{1/2}.$$

**Proof of Theorem 5**

Let $\phi_1 \geq 0$ be the first eigenvalue of the operator $A = -\Delta + q$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ normalized by $\|\phi_1\|_{L^2(\Omega)} = 1$. Then $u = \phi_1$ satisfies the assumptions of Proposition 1.

If $(\widetilde{q},\widetilde{a}) \in \mathcal{D}_m$, then

$$v = u_{\widetilde{q},\widetilde{a}}(\phi_1, i\sqrt{\lambda_1}\phi_1) - u_{q,0}(\phi_1, i\sqrt{\lambda_1}\phi_1)$$

is the solution of the following IBVP

$$\begin{cases} \partial_t^2 v - \Delta v + \widetilde{q}v + \widetilde{a}(x)\partial_t v = -[(\widetilde{q}-q) + i\sqrt{\lambda_1}\widetilde{a}]e^{i\sqrt{\lambda_1}t}\phi_1 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot,0) = 0,\ \partial_t v(\cdot,0) = 0. \end{cases}$$

From Corollary 4

$$\begin{aligned} \|\phi_1(\widetilde{q}-q)\|_{L^2(\Omega)} + \|\phi_1\widetilde{a}\|_{L^2(\Omega)} &\le C\|\partial_\nu v\|_{H^1(0,\tau;L^2(\Gamma))} \\ &\le C\|\Lambda_{\widetilde{q},\widetilde{a}} - \Lambda_{q,0}\|. \end{aligned}$$

This inequality and Proposition 1 yield

$$\|\widetilde{q}-q\|_{L^2(\Omega)} + \|\widetilde{a}-0\|_{L^2(\Omega)} \le C\|\Lambda_{\widetilde{q},\widetilde{a}} - \Lambda_{q,0}\|^{1/2}.$$

## Sketch of the proof of Proposition 1

Let $\varrho(x) = \mathrm{dist}(x, \partial\Omega)$. With help of the **strong maximum principle** one proves that $u(x) \geq c_u \varrho(x)$. Therefore

$$\int_\Omega f(x)^2 dx \leq c_u^{-1} \int_\Omega \frac{f(x)^2 u(x)^2}{\varrho(x)^2} dx.$$

Combined with **Hardy's inequality**, this estimate gives

$$\int_\Omega f(x)^2 dx \leq c_u^{-1} c \int_\Omega |\nabla(fu)(x)|^2 dx. \tag{27}$$

From the usual interpolation inequalities,

$$\|fu\|_{H^1(\Omega)} \leq C_\Omega \|fu\|_{L^2(\Omega)}^{1/2} \|fu\|_{H^2(\Omega)}^{1/2}.$$

Whence (27) implies

$$\|f\|_{L^2(\Omega)} \leq c_u \|fu\|_{L^2(\Omega)}^{1/2} \|f\|_{H^2(\Omega)}^{1/2}.$$

The stability estimate in the general case is stated in the following theorem

**Theorem 6**

*Let $(q, a) \in \mathcal{D}$. There exist $C > 0$ and $0 < \alpha < 1$, that can depend on $(q, a)$, so that for any $(\widetilde{q}, \widetilde{a}) \in \mathcal{D}$, we have*

$$\|\widetilde{q} - q\|_{L^2(\Omega)} + \|\widetilde{a} - a\|_{L^2(\Omega)} \leq C\|\Lambda_{\widetilde{q},\widetilde{a}} - \Lambda_{q,a}\|^{\alpha}.$$

*Here $\|\cdot\|$ is the norm of $\mathscr{B}(\mathcal{H}_1, H^1(0, \tau; L^2(\Gamma)))$.*

## Spectral analysis of the operator $\mathcal{A}_{q,a}$

Denote the sequence of eigenvalues, counted according to their multiplicity, of $A = -\Delta$, with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, by $0 < \lambda_1 < \lambda_2 \leq \dots \lambda_k \leq \dots$.

Define on $\mathcal{H}$

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) = \mathcal{H}_1$$

and $\mathcal{A}_{q,a} = \mathcal{A}_0 + \mathcal{B}_{q,a}$ with $D(\mathcal{A}_{q,a}) = D(\mathcal{A}_0)$, where

$$\mathcal{B}_{q,a} = \begin{pmatrix} 0 & 0 \\ -q & -a \end{pmatrix} \in \mathscr{B}(\mathcal{H}).$$

We know that $\mathcal{A}_0$ is skew-adjoint operator with $0 \in \rho(\mathcal{A}_0)$ and

$$\mathcal{A}_0^{-1} = \begin{pmatrix} 0 & -A^{-1} \\ I & 0 \end{pmatrix}.$$

As $\mathcal{A}_0^{-1} : \mathcal{H} \to \mathcal{H}_1$ is bounded and the embedding $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ is compact, $\mathcal{A}_0^{-1} : \mathcal{H} \to \mathcal{H}$ is compact.

Also, $\mathcal{A}_0$ is **diagonalizable** and its spectrum consists in the sequence $(i\sqrt{\lambda_k})$.

Consider the bounded operator $\mathcal{C}_{q,a} = (i\mathcal{A}_0^{-1})(-i\mathcal{B}_{q,a})(i\mathcal{A}_0^{-1})$. Let $s_k(\mathcal{C}_{q,a})$ be the **singular values** of $\mathcal{C}_{q,a}$, that is the eigenvalues of $(\mathcal{C}_{q,a}^*\mathcal{C}_{q,a})^{1/2}$. We have

$$s_k(\mathcal{C}_{q,a}) \leq \|\mathcal{B}_{q,a}\| s_k(i\mathcal{A}_0^{-1})^2 = \|\mathcal{B}_{q,a}\|\lambda_k^{-1},$$

where $\|\mathcal{B}_{q,a}\|$ denote the norm of $\mathcal{B}_{q,a}$ in $\mathscr{B}(\mathcal{H})$.

On the other hand, referring to **Weyl's asymptotic formula**, we have $\lambda_k^{-1} = O(k^{-2/n})$. Hence, $\mathcal{C}_{q,a}$ belongs to the **Shatten class** $\mathcal{S}_p$ for any $p > n/2$, that is

$$\sum_{k\geq 1} [s_k(\mathcal{C}_{q,a})]^p < \infty.$$

We conclude that the spectrum of $\mathcal{A}_{q,a}$ consists in a sequence of eigenvalues $(\mu_{q,a,k})$, counted according to their multiplicity, and the corresponding eigenfunctions $(\phi_{q,a,k})$ form a Riesz basis of $\mathcal{H}$.

Fix $(q, a, k)$ and set $\mu = \mu_{q,a,k}$ and $\phi = \phi_{q,a,k} = (\varphi, \psi) \in \mathcal{H}_1$ be an eigenfunction associated to $\mu$. From the classical elliptic regularity theorems, one obtains that $\varphi \in W^{2,p}(\Omega)$ for any $1 < p < \infty$.

Here again the proof of Theorem 6 is based on weighted interpolation inequality with $\phi$ as the weight function. The key point in establishing this weighted interpolation inequality is

**Proposition 2**

*There exists $\delta > 0$ so that $\varphi^{-\delta} \in L^1(\Omega)$.*

## Lemma 1

*There exists a constant $C$, that can depend on $\varphi$, so that for any $f \in L^\infty(\Omega)$, we have*

$$\|f\|_{L^2(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}^{\frac{2}{2+\delta}}\|f\varphi\|_{L^2(\Omega)}^{\frac{\delta}{2+\delta}}.$$

*Here $\delta$ is as in Proposition 2.*

Proof. Set $p = \frac{2}{\delta}$ and $\alpha = \frac{2}{p} = \frac{2\delta}{2+\delta}$. Therefore, the exponent conjugate to $p$, $p^* = \frac{2+\delta}{2}$ and $\alpha p^* = \delta$. We get by applying **Hölder's inequality**

$$\begin{aligned}\int_\Omega |f|^\alpha dx &= \int |f\varphi|^\alpha |\varphi^{-\alpha}| dx \\ &\leq \||f\varphi|^\alpha\|_{L^p(\Omega)}\||\varphi^{-\alpha}|\|_{L^{p^*}(\Omega)} = \|f\varphi\|_{L^2(\Omega)}^{\frac{2}{p}}\|\varphi^{-\delta}\|_{L^1(\Omega)}^{1/p^*}.\end{aligned}$$

Whence

$$\|f\|_{L^\alpha(\Omega)} \leq \|f\varphi\|_{L^2(\Omega)}\|\varphi^{-\delta}\|_{L^1(\Omega)}^{1/\delta}. \tag{28}$$

On the other hand

$$\|f\|_{L^2(\Omega)} \leq \|f\|_{L^\infty(\Omega)}^{\frac{2-\alpha}{2}} \|f\|_{L^\alpha(\Omega)}^{\frac{\alpha}{2}}. \tag{29}$$

A combination of (28) and (29) yields

$$\|f\|_{L^2(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}^{\frac{2}{2+\delta}} \|f\varphi\|_{L^2(\Omega)}^{\frac{\delta}{2+\delta}}.$$

**Completion of the proof of Theorem 6**

We proceed as in the proof of Theorem 5 with $(u_0, u_1) = \phi_{q,a,k}$. We obtain

$$\|\varphi(\widetilde{q} - q)\|_{L^2(\Omega)} + \|\varphi(\widetilde{a} - a)\|_{L^2(\Omega)} \leq C\|\Lambda_{\widetilde{q},\widetilde{a}} - \Lambda_{q,a}\|.$$

In light of the weighted interpolation inequality in Lemma 1, this inequality entails

$$\|\widetilde{q} - q\|_{L^2(\Omega)} + \|\widetilde{a} - a\|_{L^2(\Omega)} \leq C\|\Lambda_{\widetilde{q},\widetilde{a}} - \Lambda_{q,a}\|^{\frac{\delta}{2+\delta}}.$$

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In this section $\Omega = (0,1)$.

Introduce the notations

$$\begin{aligned} H_0 &= L^2(0,1), \\ H_{1/2} &= H_0^2(0,1), \\ H_1 &= H^4(0,1) \cap H_0^2(0,1). \end{aligned}$$

Define on $\mathcal{H} := H_{1/2} \oplus H_0$

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -\frac{d^4}{dx^4} & 0 \end{pmatrix}, \quad D(\mathcal{A}) = H_1 \oplus H_{1/2} := \mathcal{H}_1$$

and consider the observation operator $C : \mathcal{H}_1 \to \mathbb{C}$ given by

$$C \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{d^2 \varphi}{dx^2}(0).$$

Consider the following IBVP

$$\begin{cases} \partial_t^2 u + \partial_x^4 u = 0 & \text{in } Q, \\ u(0,\cdot) = u(1,\cdot) = 0 & \text{on } (0,\tau), \\ \partial_x u(0,\cdot) = \partial_x u(1,\cdot) = 0 & \text{on } (0,\tau), \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1. \end{cases} \tag{30}$$

The operator $\mathcal{A}$ is skew-adjoint and therefore it generates a unitary group on $\mathcal{H}$. Consequently, for any $(u_0,u_1) \in \mathcal{H}_1$ the IBVP (30) has a unique solution $u$ so that $(u,\partial_t u) \in C([0,\tau],\mathcal{H}_1) \cap C^1([0,\tau],\mathcal{H})$. Moreover, $(\mathcal{A},\mathcal{C})$ is exactly observable for any $\tau > 0$ and there is a constant $\kappa > 0$ such that

$$\kappa^2 \|(u_0,u_1)\|_{\mathcal{H}} \le \|\partial_x^2 u(0,\cdot)\|^2_{L^2(0,\tau)}. \tag{31}$$

Here the constant $\kappa$ is independent on $(u_0,u_1)$.

Let $\mathcal{B}_a$ be the operator, where $a = a(x)$,

$$\mathcal{B}_a = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}.$$

This operator is bounded on $\mathcal{H}$ whenever $a \in L^\infty(\Omega)$. Therefore, $\mathcal{A} + \mathcal{B}_a$ generates a continuous semigroup and then the IBVP

$$\begin{cases} \partial_t^2 u + \partial_x^4 u + a(x)\partial_t u = 0 & \text{in } Q, \\ u(0,\cdot) = u(1,\cdot) = 0 & \text{on } (0,\tau), \\ \partial_x u(0,\cdot) = \partial_x u(1,\cdot) = 0 & \text{on } (0,\tau), \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1 & \end{cases} \tag{32}$$

has a unique solution $u_a = u_a(u_0,u_1)$ satisfying $(u_a, \partial_t u_a) \in C([0,T],\mathcal{H}_1) \cap C^1([0,T],\mathcal{H})$, for any $(u_0,u_1) \in \mathcal{H}_1$. Moreover, by a perturbation argument, there exists $\delta > 0$ so that $(\mathcal{A} + \mathcal{B}_a, \mathcal{C})$ is exactly observable with constant $\geq \kappa^2/2$ if $\|\mathcal{B}_a\|_{\mathscr{B}(\mathcal{H})} \leq \delta$. That is to say

$$(1/2)\kappa^2 \|(u_0,u_1)\|_{\mathcal{H}} \leq \|\partial_x^2 u(0,\cdot)\|^2_{L^2(0,\tau)}. \tag{33}$$

Recall that the spectrum of $\mathcal{A}$ consists in a sequence of simple eigenvalues $(i\rho_k)_{k\in\mathbb{Z}^*}$, where

$$\rho_k = \pi^2\left(k-\frac{1}{2}\right)^2 + a_k, \quad k\in\mathbb{N}^*,$$

$(a_k)$ a sequence converging exponentially to 0, and $\rho_{-k}=-\rho_k$, $k\in\mathbb{N}^*$.

Let $A_0=\frac{d^4}{dx^4}$ with domain $D(A_0)=H^4(\Omega)\cap H_0^2(\Omega)$. Then $A_0$ is diagonalizable with eigenvalues $(\rho_k^2)_{k\in\mathbb{N}^*}$. Let $(f_k)_{k\in\mathbb{N}^*}$ be a basis of eigenfunctions, each $f_k$ corresponding to $\rho_k^2$. Let

$$g_k=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{f_k}{i\rho_k}\\ f_k\end{pmatrix}, \text{ and } g_{-k}=-g_k, \quad k\in\mathbb{N}^*.$$

Then $(g_k)_{k\in\mathbb{Z}^*}$ forms an orthonormal basis of $\mathcal{A}$.

Define $\mathcal{H}_\pm$ as the closure of $\mathrm{span}\{g_{\pm k};\ k \in \mathbb{N}^*\}$. Then $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ and $\mathcal{H}_\pm$ is invariant under $\mathcal{A}$. We consider $\mathcal{A}^\pm : \mathcal{H}_\pm \to \mathcal{H}_\pm$ the operator given by $\mathcal{A}^\pm = \mathcal{A}_{|\mathcal{H}_\pm}$ and

$$D(\mathcal{A}^\pm) = \{u \in \mathcal{H}_\pm;\ \sum_{k\in\mathbb{N}^*} k^4 |\langle u, g_{\pm k}\rangle|^2 < \infty\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{H}$, and we set $\mathcal{A}^\pm_{a_0} = \mathcal{A}^\pm + \mathcal{B}_{a_0}$. Since $\rho_{k+1} - \rho_k \to +\infty$ as $k \to +\infty$, $(\rho_k)_{k\in\mathbb{N}^*}$ satisfies the a gap condition. Precisely, there exists $d > 0$ so that

$$\rho_{k+1} - \rho_k \geq d, \quad k \in \mathbb{N}^*.$$

Set

$$\varrho = \sum_{k\geq 1} \frac{1}{(2k+1)^2} \ \text{ and } \ \alpha = \frac{d}{2\sqrt{2(1+\varrho)}}.$$

## Theorem 7

*Under the assumption*

$$\rho := \|a_0\|_\infty < \alpha,$$

*the spectrum of* $\pm\mathcal{A}^\pm_{a_0}$ *consists in a sequence* $(i\mu_k^\pm)$ *such that, for any* $\delta \in (0, 1 - \rho^2/\alpha^2)$, *there is an integer* $\widetilde{k}$ *such that*

$$|i\mu_k^\pm - i\rho_k| \leq \overline{\alpha} = \overline{\alpha}(a_0) := \frac{\rho d}{\sqrt{4\rho^2 + d^2\delta}}, \quad k \geq \widetilde{k}. \tag{34}$$

*In addition,* $\mathcal{H}^\pm$ *admits a Riesz basis* $(\phi_k^\pm) = ((\varphi_k^\pm, i\mu_k^\pm\varphi_k^\pm))$, *each* $\phi_k^\pm$ *is an eigenfunction corresponding to* $i\mu_k^\pm$.

Denote by $(\widetilde{\phi}_k^{\pm})$ the Riesz basis biorthogonal to $(\phi_k^{\pm})$ and define the sequence $(\phi_k)_{k\in\mathbb{Z}^*}$ (resp. $(\widetilde{\phi}_k)_{k\in\mathbb{Z}^*}$) as follows $\phi_{-k}=-\phi_k^-$ and $\phi_k=\phi_k^+$ (resp. $\widetilde{\phi}_{-k}=-\widetilde{\phi}_k^-$ and $\widetilde{\phi}_k=\widetilde{\phi}_k^+$), $k\in\mathbb{N}^*$. Set also $\mu_{-k}=-\mu_k^-$ and $\mu_k=\mu_k^+$, $k\in\mathbb{N}^*$. Therefore, $\mathcal{A}_{a_0}\phi_k=i\mu_k\phi_k$, $k\in\mathbb{Z}^*$, and, for any $u\in\mathcal{H}$,

$$u=\sum_{k\in\mathbb{Z}^*}\langle u,\widetilde{\phi}_k\rangle\phi_k=\sum_{k\in\mathbb{Z}^*}\langle u,\phi_k\rangle\widetilde{\phi}_k.$$

Additionally,

$$\alpha\|u\|_{\mathcal{H}}^2\le\sum_{k\in\mathbb{Z}^*}|\langle u,\widetilde{\phi}_k\rangle|^2,\ \sum_{k\in\mathbb{Z}^*}|\langle u,\phi_k\rangle|^2\le\beta\|u\|_{\mathcal{H}}^2, \tag{35}$$

where the constants $\alpha$ and $\beta$ do not depend on $u$.

Pick $a_0$ as in Theorem 7. Then $u = u_a - u_{a_0}$ is the solution of the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = (a_0 - a) i\mu_k e^{i\mu_k t}\varphi_k & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = 0. & \end{cases} \tag{36}$$

Define the initial-to-boundary operator $\Lambda_a$ by

$$\Lambda_a : u_1 \in H_{1/2} \longrightarrow \partial_x^2 u_a(0, u_1)(0,\cdot) \in L^2(0,\tau).$$

Using estimate (34) and the stability estimate for the inverse source problem, we get

$$|(a - a_0, \psi_k)|^2 = |\langle (0, a - a_0), \phi_k\rangle|^2 \leq Ce^{Ck^4} \|\Lambda_a - \Lambda_{a_0}\|^2. \tag{37}$$

It follows from (35),

$$\alpha \|a - a_0\|_2^2 = \alpha \left\|(0, a - a_0)\right\|_{\mathcal{H}}^2 \leq \sum_{|k|\geq 1} |\langle (0, a - a_0), \phi_k\rangle|^2. \tag{38}$$

In light of (37) and (38), we have

$$\begin{aligned}\alpha\|a-a_0\|_2^2 &\leq CNe^{C\lambda}\|\Lambda_a-\Lambda_{a_0}\|^2+\frac{1}{\lambda}\sum_{|k|>N}k^4|(a-a_0,\psi_k)|^2\\ &\leq CNe^{C\lambda}\|\Lambda_a-\Lambda_{a_0}\|^2+\frac{1}{\lambda}\sum_{|k|\geq 1}k^4|(a-a_0,\psi_k)|^2. \qquad (39)\end{aligned}$$

Here $\lambda \geq \lambda_1$ and $N = N(\lambda)$ is the integer so that $N \leq \lambda^{1/4} < N+1$.
We make the assumption, where $m > 0$ is fixed,

$$\sum_{|k|\geq 1}k^4|(a-a_0,\psi_k)|^2 \leq m. \qquad (40)$$

Under this assumption, (39) entails

$$\alpha\|a-a_0\|_2^2 \leq Ce^{C\lambda}\|\Lambda_a-\Lambda_{a_0}\|^2+\frac{m}{\lambda}.$$

A minimization argument allows us to obtain

**Theorem 8**

*There exist two constants $C > 0$ and $\delta > 0$ so that*

$$C\|a - a_0\|_2 \leq \left|\ln \|\Lambda_a - \Lambda_{a_0}\|\right|^{-1/2} + \|\Lambda_a - \Lambda_{a_0}\|,$$

*for any $a \in L^\infty(\Omega)$ so that (40) holds.*

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In this section $\Gamma$ is an arbitrary non empty open subset of $\partial\Omega$.

Recall that $\mathcal{H} = H_0^1(\Omega) \oplus L^2(\Omega)$.

Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = u_0, \; \partial_t u(\cdot, 0) = u_1. \end{cases} \tag{41}$$

As we have seen before, for each $(u_0, u_1) \in \mathcal{H}$ and $q \in L^\infty(\Omega)$, the IBVP (41) has a unique solution $u_q = u_q(u_0, u_1) \in C([0, \tau]; H_0^1(\Omega))$ so that $\partial_t u_q \in C([0, \tau]; L^2(\Omega))$ and $\partial_\nu u_q \in L^2(\Sigma)$.

Additionally, for any $m > 0$, there exists a constant $C = C(m, \Omega) > 0$ so that, for each $q \in mB_{L^\infty(\Omega)}$ and $(u_0, u_1) \in \mathcal{H}$,

$$\|\partial_\nu u_q(u_0, u_1)\|_{L^2(\Sigma)} \leq C\|(u_0, u_1)\|_{\mathcal{H}}. \tag{42}$$

Therefore, the mapping

$$\Lambda_q : u_0 \to \partial_\nu u_q(u_0, 0)_{|\Gamma\times(0,\tau)}$$

defines a bounded operator from $H = H^2(\Omega) \cap H_0^1(\Omega)$ into $H^1(0, \tau; L^2(\Gamma))$ and, for any $q \in mB_{L^\infty(\Omega)}$,

$$\|\Lambda_q\| \leq C,$$

where $C$ is the same constant as in (42).

Here and henceforth, $\|\cdot\|$ is the norm of $\mathscr{B}(H, H^1(0, \tau; L^2(\Gamma)))$.

Let

$$\Psi(\gamma) = |\ln\gamma|^{-\frac{1}{8+2n}} + \gamma, \;\; \gamma > 0,$$

extended by continuity at $\gamma = 0$ by setting $\Psi(0) = 0$.

### Theorem 9

*There exist two constants $\tau_0 > 0$ and $C > 0$ so that, for any $\tau \geq \tau_0$, $q_0, q \in mB_1$ satisfying $q_0 \geq 0$ and $q - q_0 \in mB_{W^{1,\infty}(\Omega)}$,*

$$C\|q_0 - q\|_{L^2(\Omega)} \leq \Psi\left(\|\Lambda_q - \Lambda_{q_0}\|\right).$$

In the present case we cannot use an observability inequality, instead we appeal to the following result :

### Theorem 10 (Robbiano, AA '95)

*There exist three constants $\tau_0 > 0$, $C > 0$ and $\mu > 0$ so that, for all $\tau \geq \tau_0$, $(u_0, u_1) \in \mathcal{H}$, $q \in mB_{L^\infty(\Omega)}$ and $\epsilon > 0$,*

$$C\|(u_0, u_1)\|_{\mathcal{H}_{-1}} \leq \frac{1}{\sqrt{\epsilon}}\|(u_0, u_1)\|_{\mathcal{H}} + e^{\mu\epsilon}\|\partial_\nu u_q(u_0, u_1)\|_{L^2(\Gamma\times(0,\tau))},$$

*where $\mathcal{H}_{-1} = L^2(\Omega) \oplus H^{-1}(\Omega)$.*

## Sketch of the proof of Theorem 9

Let $A_0 : L^2(\Omega) \to L^2(\Omega)$ the unbounded given by

$$A_0 = -\Delta + q_0, \quad D(A_0) = H_0^1(\Omega) \cap H^2(\Omega).$$

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots \to +\infty$ be the sequence of eigenvalues of the operator $A_0$ and $(\phi_k)$ a sequence of the corresponding eigenfunctions so that $(\phi_k)$ forms an orthonormal basis of $L^2(\Omega)$.

Since $v = u_q(\phi_k) - u_{q_0}(\phi_k)$ is the solution of the IBVP

$$\begin{cases} \partial_t^2 v - \Delta v + q(x)v = \cos(\sqrt{\lambda_k} t)(q - q_0)\phi_k & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot, 0) = 0, \ \partial_t v(\cdot, 0) = 0, \end{cases}$$

we get by applying the stability estimate for the inverse source problem together with Theorem 10 :

$$C\|(q-q_0)\phi_k\|_{H^{-1}(\Omega)} \leq \frac{1}{\sqrt{\epsilon}}\|(q-q_0)\phi_k\|_{L^2(\Omega)} + e^{\tau^2\lambda_k}e^{\mu\epsilon}\|\Lambda_q(\phi_k)-\Lambda_{q_0}(\phi_k)\|_{H^1(0,\tau;L^2(\Sigma))}.$$

By straightforward computation one obtains

$$C\|(q-q_0)\phi_k\|_{H^{-1}(\Omega)} \leq \frac{1}{\sqrt{\epsilon}} + e^{\kappa\lambda_k}e^{\mu\epsilon}\|\Lambda_q-\Lambda_{q_0}\|,$$

which, combined with an interpolation inequality, leads

$$C\|(q-q_0)\phi_k\|^2_{L^2(\Omega)} \leq \frac{\lambda_k}{\sqrt{\epsilon}} + e^{(\kappa+1)\lambda_k}e^{\mu\epsilon}\|\Lambda_q-\Lambda_{q_0}\|.$$

Therefore

$$C(q-q_0,\phi_k)^2_{L^2(\Omega)} \leq \frac{\lambda_k}{\sqrt{\epsilon}} + e^{(\kappa+1)\lambda_k}e^{\mu\epsilon}\|\Lambda_q-\Lambda_{q_0}\|. \tag{43}$$

On the other hand

$$\|q-q_0\|^2_{L^2(\Omega)} \le \sum_{k\le N} (q-q_0,\phi_k)^2_{L^2(\Omega)} + \frac{cm^2}{(N+1)^{2/n}}.$$

Whence

$$C\|q-q_0\|^2_{L^2(\Omega)} \le \frac{N^{1+2/n}}{\sqrt{\epsilon}} + \frac{1}{(N+1)^{2/n}} + Ne^{\varrho N^{2/n}} e^{\mu\epsilon}\|\Lambda_q - \Lambda_{q_0}\|, \quad (44)$$

where we used that $\lambda_k = O(k^{2/n})$. Equivalently

$$C\|q-q_0\|^2_{L^2(\Omega)} \le \frac{s^{1+2/n}}{\sqrt{\epsilon}} + \frac{1}{s^{2/n}} + se^{\varrho s^{2/n}} e^{\mu\epsilon}\|\Lambda_q - \Lambda_{q_0}\|, \quad s \ge 1.$$

$\epsilon = s^{8/n+2}$ in this inequality gives

$$C\|q-q_0\|^2_{L^2(\Omega)} \le \frac{1}{s^{2/n}} + e^{cs^{8/n+2}}\|\Lambda_q - \Lambda_{q_0}\|.$$

The expected inequality follows then by minimizing with respect to $s$.

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## Inverse source problem

Consider the IBVP

$$\begin{cases} \partial_t u - \Delta u + q(x)u = g(t)f(x) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot,0) = 0. \end{cases} \tag{45}$$

Recall that $H^{2,1}(Q) = L^2((0,\tau),H^2(\Omega)) \cap H^1((0,\tau),L^2(\Omega))$.

For any $(f,g) \in L^2(\Omega) \oplus L^2(0,\tau)$ and $q \in L^\infty(\Omega)$, the IBVP (45) has a unique solution $u_q(f,g) \in H^{2,1}(Q)$.

In the sequel, $\Gamma$ is nonempty open subset of $\partial\Omega$.

When $(f,g) \in L^2(\Omega) \oplus H^1(0,\tau)$, $\partial_\nu u_q(f,g)$ is well defined as an element of $H^1(0,\tau;L^2(\Sigma))$ and

$$\|\partial_\nu u_q(f,g)\|_{H^1(0,\tau;L^2(\Sigma))} \leq C\|(f,g)\|_{L^2(\Omega)\oplus H^1(0,\tau)},$$

uniformly in $q \in mB_{L^\infty(\Omega)}$.

Denote by $v_q(f) \in H^{2,1}(Q)$ the solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot, 0) = f. \end{cases}$$

From the identity

$$\partial_\nu u_q(f,g)_{|\Gamma}(\cdot, t) = \int_0^t g(t-s)\partial_\nu v_q(f)_{|\Gamma}(\cdot, s)ds,$$

the final time observability inequality and the stability estimate for the abstract inverse source problem, we get

$$C\|v_q(f)(\cdot,\tau)\|_{L^2(\Omega)} \leq \frac{1}{|g(0)|}e^{\tau\frac{\|g'\|^2_{L^2((0,\tau))}}{|g(0)|^2}}\|\partial_\nu u_q(f,g)\|_{H^1(0,\tau;L^2(\Gamma))}. \quad (46)$$

Denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots \to +\infty$ the sequence of eigenvalues of the operator

$$A = -\Delta + q, \quad D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Let $(\phi_k)$ be a sequence of the corresponding eigenfunctions so that $(\phi_k)$ forms an orthonormal basis of $L^2(\Omega)$.

Then

$$v_q(f)(\cdot,\tau) = \sum_{\ell \geq 1} e^{-\lambda_\ell \tau}(f, \phi_\ell)\phi_\ell.$$

Similar computations to that carried out in Section 1 yield

$$\|f\|^2_{L^2(\Omega)} \leq e^{c\epsilon}\|v_q(f)(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{C}{\epsilon}\|f\|^2_{H^1(\Omega)}, \quad \epsilon \geq 1. \tag{47}$$

A combination of (46) and (47) yields

## Theorem 11

*There exist two constants $c > 0$ and $C > 0$ so that, for any $q \in mB_{L^\infty(\Omega)}$, $f \in H_0^1(\Omega)$ and $g \in H^1(0,\tau)$ with $g(0) \neq 0$,*

$$C\|f\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\epsilon}}\|f\|_{H_0^1(\Omega)} + \frac{1}{|g(0)|} e^{\tau \frac{\|g'\|^2_{L^2(0,\tau)}}{|g(0)|^2}} e^{c\epsilon}\|\partial_\nu u_q(f,g)\|_{H^1(0,\tau;L^2(\Gamma))}, \quad \epsilon \geq 1 \tag{48}$$

An immediate consequence is

**Corollary 12**

*Fix $q \in L^\infty(\Omega)$ and $g \in H^1(0,\tau)$ satisfying $g(0) \neq 0$. There exists a constant $C = C(n,\Omega,q,g,\Gamma,m) > 0$ so that, for any $f \in mB_{H_0^1(\Omega)}$,*

$$C\|f\|_{L^2(\Omega)} \leq \Phi\left(\|\partial_\nu u_q(f,g)\|_{H^1(0,\tau;L^2(\Gamma))}\right),$$

*where $\Phi(\gamma) = |\ln\gamma|^{-1/2} + \gamma$, $\gamma > 0$.*

## Determining the zero order term

Consider the IBVP

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot,0) = u_0. \end{cases} \tag{49}$$

Let $\mathcal{H}_0 = \{w \in H_0^1(\Omega);\ \Delta w \in H_0^1(\Omega)\}$ that we equip with its natural norm

$$\|u\|_{\mathcal{H}_0} = \|u\|_{H_0^1(\Omega)} + \|\Delta u\|_{H_0^1(\Omega)}.$$

We show that if $u_0 \in \mathcal{H}_0$ and $q \in W^{1,\infty}(\Omega)$, the IBVP (49) has a unique solution $u_q(u_0) \in H^{2,1}(Q)$ so that $\partial_t u \in H^{2,1}(Q)$ and

$$\|u_q(u_0)\|_{H^{2,1}(Q)} + \|\partial_t u_q(u_0)\|_{H^{2,1}(Q)} \leq C\|u\|_{\mathcal{H}_0},$$

uniformly in $q \in mB_{W^{1,\infty}(\Omega)}$.

Therefore $\mathcal{N}_q : u_0 \in \mathcal{H}_0 \mapsto \partial_\nu u_q(u_0) \in H^1((0,\tau); L^2(\Gamma))$ is bounded and

$$\|\mathcal{N}_q\| \leq C,$$

uniformly in $q \in mB_{W^{1,\infty}(\Omega)}$. Here and henceforth, $\|\cdot\|$ denotes the norm of $\mathscr{B}(\mathcal{H}_0, H^1(0,\tau; L^2(\Gamma)))$.

**Theorem 13**

*There exists a constant $C > 0$ so that, for any $q_0$, $q \in mB_{W^{1,\infty}(\Omega)}$,*

$$C\|q - q_0\|_{L^2(\Omega)} \leq \Theta\left(\|\mathcal{N}_q - \mathcal{N}_{q_0}\|\right).$$

*Here $\Theta(\gamma) = |\ln \gamma|^{-\frac{1}{n+4}} + \gamma$.*

## Sketch of the proof

Let $A_0 = -\Delta + q_0$ with domain $D(A_0) = H_0^1(\Omega) \cap H^2(\Omega)$. Denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots \to +\infty$ the sequence of eigenvalues of the operator $A_0$, and $(\phi_k)$ a sequence of the corresponding eigenfunctions so that $(\phi_k)$ form an orthonormal basis of $L^2(\Omega)$.

Since $v = u_q(\phi_k) - u_{q_0}(\phi_k)$ is the solution of the IBVP

$$\begin{cases} \partial_t v - \Delta v + q(x)v = (q - q_0)\phi_k e^{-\lambda_k t} & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = u_0. \end{cases}$$

we get from (48)

$$C|(q-q_0,\phi_k)| \leq \frac{\sqrt{\lambda_k}}{\sqrt{\epsilon}} + e^{\tau\lambda_k^2} e^{c\epsilon}\lambda_k^2\|\mathcal{N}_q - \mathcal{N}_{q_0}\|. \tag{50}$$

A straightforward consequence of estimate (50) is

$$C\sum_{k=1}^{N}|(q-q_0,\phi_k)|^2 \leq \frac{N\lambda_N}{\epsilon} + Ne^{(2\tau+1)\lambda_N^2}e^{c\epsilon}\|\mathcal{N}_q - \mathcal{N}_{q_0}\|^2, \tag{51}$$

for any arbitrary integer $N \geq 1$.
We pursue similarly to the proof of Theorem 9 in order to get, for arbitrary $s \geq 1$,

$$C\|q-q_0\|_{L^2(\Omega)}^2 \leq \frac{s^{1+2/n}}{\epsilon} + \frac{1}{s^{2/n}} + e^{\varrho s^{1+4/n}}e^{c\epsilon}\|\mathcal{N}_q - \mathcal{N}_{q_0}\|^2.$$

The proof is then completed in the same manner to that of Theorem 9.

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Here $\Omega = (0,1) \times (0,1)$. Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma_0, \\ \partial_\nu u + a(x,y)\partial_t u = 0 & \text{on } \Sigma_1, \\ u(\cdot,0) = u^0,\ \partial_t u(\cdot,0) = u^1. \end{cases} \tag{52}$$

Here $\Sigma_0 = \Gamma_0 \times (0,\tau)$, $\Sigma_1 = \Gamma_1 \times (0,\tau)$, where

$$\Gamma_0 = ((0,1) \times \{1\}) \cup (\{1\} \times (0,1)),$$
$$\Gamma_1 = ((0,1) \times \{0\}) \cup (\{0\} \times (0,1))$$

We identify $a_{|(0,1)\times\{0\}}$ by $a_1 = a_1(x)$, $x \in (0,1)$ and $a_{|\{0\}\times(0,1)}$ by $a_2 = a_2(y)$, $y \in (0,1)$. In that case it is natural to identify $a$, defined on $\Gamma_1$, by the pair $(a_1, a_2)$.

Fix $1/2 < \alpha \leq 1$ and assume that $a \in \mathscr{A}$, where

$$\mathscr{A} = \{b = (b_1, b_2) \in C^\alpha[0,1] \oplus C^\alpha[0,1],\ b_1(0) = b_2(0),\ b_j \geq 0\}.$$

Note that in this case that the multiplication by $a_j$, $j = 1, 2$, defines a bounded operator on $H^{1/2}(0,1)$.

Let $V = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_0\}$ and consider

$$A_a = (w, \Delta v), \quad D(A_a) = \{(v, w) \in V \oplus V; \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = -aw \text{ on } \Gamma_1\}.$$

The operator $A_a$ is a skew-adjoint on $V \oplus L^2(\Omega)$. Therefore, $A_a$ is the generator of a strongly continuous group of contractions $e^{tA_a}$. Whence, for each $(u^0, u^1) \in D(A_a)$, the IBVP (52) possesses a unique solution denoted by $u_a = u_a(u^0, u^1)$ so that

$$(u_a, \partial_t u_a) \in C([0, \infty); D(A_a)) \cap C^1([0, \infty), V \oplus L^2(\Omega)).$$

For $0 < m \le M$, we set

$$\mathscr{A}_{m,M} = \{b = (b_1, b_2) \in \mathscr{A} \cap H^1(0,1) \oplus H^1(0,1);\ m \le b_j,\ \|b_j\|^2_{H^1(0,1)} \le M\}.$$

Let $\mathcal{U}_0$ given by

$$\mathcal{U}_0 = \{v \in V;\ \Delta v \in L^2(\Omega) \text{ and } \partial_\nu v = 0 \text{ on } \Gamma_1\}.$$

Observe then that $\mathcal{U}_0 \times \{0\} \subset D(A_a)$, for any $a \in \mathscr{A}$.
Let $C_a \in \mathscr{B}(D(A_a); L^2(\Sigma_1))$ given by

$$C_a(u^0, u^1) = \partial_\nu u_a(u^0, u^1)_{|\Gamma_1}.$$

Define the initial-to-boundary operator

$$\Lambda_a : u^0 \in \mathcal{U}_0 \longrightarrow C_a(u^0, 0) \in L^2(\Sigma_1).$$

Then $\Lambda_a \in \mathscr{B}(\mathcal{U}_0; L^2(\Sigma_1))$, when $\mathcal{U}_0$ is endowed with norm

$$\|u^0\|_{\mathcal{U}_0} = \left(\|u^0\|_V^2 + \|\Delta u^0\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Henceforth, the norm of $\mathscr{B}(\mathcal{U}_0; L^2(\Sigma_1))$ is denoted by $\|\cdot\|$.

**Theorem 14**

*There exists $\tau_0 > 0$ so that for any $\tau > \tau_0$, we find a constant $c > 0$ depending only on $\tau$ such that*

$$\|a - 0\|_{L^2(0,1)\oplus L^2(0,1)} \leq cM\left(\left|\ln\left(m^{-1}\|\Lambda_a - \Lambda_0\|\right)\right|^{-1/2} + m^{-1}\|\Lambda_a - \Lambda_0\|\right), \tag{53}$$

*for each $a \in \mathcal{A}_{m,M}$.*

## Observability inequality

Note that

$$\Gamma_0 \subset \{x \in \Gamma;\ m(x) \cdot \nu(x) < 0\},$$
$$\Gamma_1 \subset \{x \in \Gamma;\ m(x) \cdot \nu(x) > 0\},$$

where $m(x) = x - x_0$, $x \in \mathbb{R}^2$, and $x_0 = (s, s)$ with $s > 1$.
Recall the following **Rellich identity** : for each $3/2 < s < 2$ and $\varphi \in H^s(\Omega)$ satisfying $\Delta\varphi \in L^2(\Omega)$,

$$2\int_\Omega \Delta\varphi(m \cdot \nabla\varphi)dx = 2\int_\Gamma \partial_\nu\varphi(m \cdot \nabla\varphi)d\sigma - \int_\Gamma (m \cdot \nu)|\nabla\varphi|^2 d\sigma. \quad (54)$$

**Lemma 15**

*Let $(v, w) \in D(A_a)$. Then*

$$2\int_\Omega \Delta v(m \cdot \nabla v)dx = 2\int_\Gamma \partial_\nu v(m \cdot \nabla v)d\sigma - \int_\Gamma (m \cdot \nu)|\nabla v|^2 d\sigma.$$

## Sketch of the proof

Let $(v, w) \in D(A_a)$. By an appropriate extension theorem, there exists $\widetilde{v} \in H^2(\Omega)$ so that $\widetilde{v} = 0$ on $\Gamma_0$ and $\partial_\nu \widetilde{v} = -aw$ on $\Gamma_1$.

In light of the fact that $z = v - \widetilde{v}$ is such that $\Delta z \in L^2(\Omega)$, $z = 0$ on $\Gamma_0$ and $\partial_\nu z = 0$ on $\Gamma_1$, we get $z \in H^s(\Omega)$ for some $3/2 < s < 2$. Therefore $v \in H^s(\Omega)$.

We complete the proof by applying Rellich identity (54).

With the help of Lemma 15, we prove

**Theorem 16**

*Assume that $a \geq \delta$ on $\Gamma_1$, for some $\delta > 0$. There exist $M \geq 1$ and $\omega > 0$, depending only on $\delta$, so that*

$$\|e^{tA_a}(v, w)\|_{V \oplus L^2(\Omega)} \leq Me^{-\omega t}\|(v, w)\|_{V \oplus L^2(\Omega)}, \quad (v, w) \in D(A_a),\ t \geq 0.$$

A standard argument leads

**Corollary 17**

*Fix $0 < \delta_0 < \delta_1$. Then there exist $\tau_0 > 0$ and $\kappa$, depending only on $\delta_0$ and $\delta_1$ so that for any $\tau \geq \tau_0$ and $a \in \mathscr{A}$ satisfying $\delta_0 \leq a \leq \delta_1$ on $\Gamma_1$,*

$$\kappa \|(u^0, u^1)\|_{V \oplus L^2(\Omega)} \leq \|C_a(u^0, u^1)\|_{L^2(\Sigma_1)}.$$

*Moreover, $C_a$ is admissible for $e^{tA_a}$ and $(C_a, A_a)$ is exactly observable.*

## Inverse source problem with singular source

Let $H$ be a Hilbert space, $A : D(A) \subset H \to H$ be the generator of continuous semigroup $(T(t))$ and $C \in \mathscr{B}(D(A), Y)$, $Y$ is a Hilbert space which is identified with its dual space an admissible observation for $(T(t))$. Define $\Psi \in \mathscr{B}(D(A), L^2((0,\tau), Y))$ by

$$(\Psi x)(t) = CT(t)x, \quad t \in [0, \tau], \quad x \in D(A),$$

Assume that $(A, C)$ is exactly observable at time $\tau > 0$. That is there exists a constant $\kappa$ so that

$$\int_0^\tau \|(\Psi x)(t)\|_Y^2 dt \geq \kappa^2 \|x\|_X^2, \quad x \in D(A).$$

Let $\lambda \in H^1(0, \tau)$ with $\lambda(0) \neq 0$. Consider the Cauchy problem

$$z'(t) = Az(t) + \lambda(t)x, \quad z(0) = 0 \tag{55}$$

and set

$$y(t) = Cz(t), \quad t \in [0, \tau]. \tag{56}$$

Fix $\beta \in \rho(A)$. Let $H_1 := D(A)$ equipped with the norm $\|x\|_1 = \|(\beta - A)x\|$ and denote by $H_{-1}$ the completion of $H$ with respect to the norm $\|x\|_{-1} = \|(\beta - A)^{-1}x\|$.

According to the classical **extrapolation theory of semigroups**, for any $x \in H_{-1}$, the Cauchy problem (18) has a unique solution $z \in C([0, \tau]; H)$. Moreover $y$ given in (19) belongs to $L^2(0, \tau; Y)$.

By Duhamel's formula, where $x \in H$,

$$y(t) = \int_0^t \lambda(t-s)CT(s)x ds = \int_0^t \lambda(t-s)(\Psi x)(s) ds. \qquad (57)$$

Let

$$H^1_\ell(0,\tau;Y) = \left\{u \in H^1(0,\tau;Y);\ u(0) = 0\right\}.$$

Define the operator $S : L^2(0,\tau;Y) \longrightarrow H^1_\ell(0,\tau;Y)$ by

$$(Sh)(t) = \int_0^t \lambda(t-s)h(s) ds. \qquad (58)$$

If $E = S\Psi$, then (57) takes the form

$$y(t) = (Ex)(t).$$

Let $\mathcal{Z} = (\beta - A^*)^{-1}(X + C^* Y)$.

## Theorem 18

*Assume that $(A, C)$ is exactly observable at time $\tau$. Then*
*(i) $E$ is one-to-one from $H$ onto $H^1_\ell(0, \tau; Y)$.*
*(ii) $E$ can be extended to an isomorphism, denoted by $\widetilde{E}$, from $\mathcal{Z}'$ onto $L^2(0, \tau; Y)$.*
*(iii) There exists a constant $\widetilde{\kappa}$, independent on $\lambda$, so that*

$$\|x\|_{\mathcal{Z}'} \leq \widetilde{\kappa}|\lambda(0)|e^{\frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}\tau}\|\widetilde{E}x\|_{L^2(0,\tau;Y)}. \tag{59}$$

Consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta u = \lambda(t) w & \text{in } Q, \\ u = 0 & \text{on } \Sigma_0, \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \Sigma_1, \\ u(\cdot,0) = 0,\ \partial_t u(\cdot,0) = 0, \end{cases} \tag{60}$$

We are going to apply the preceding abstract result when $H = V \oplus L^2(\Omega)$, $H_1 = D(A_a)$ equipped with its **graph norm** and $Y = L^2(\Gamma_1)$.

If $(0, w) \in H_{-1}$ and $\lambda \in H^1(0,\tau)$, the IBVP (60) has a unique solution $u_w$ so that $u_w, \partial_t u_w \in C([0,\tau]; V \oplus L^2(\Omega))$ and $\partial_\nu u_{w|\Gamma_1} \in L^2(\Sigma_1)$.

Since $\{0\} \times V' \subset H_{-1}$, we obtain as a consequence of Corollary 17 and Theorem 18

## Proposition 3

*There exists a constant $C > 0$ so that for any $\lambda \in H^1(0,\tau)$, with $\lambda(0) \neq 0$, and $w \in V'$,*

$$\|w\|_{V'} \leq C|\lambda(0)|e^{\frac{\|\lambda'\|^2_{L^2(0,\tau)}}{|\lambda(0)|^2}\tau}\|\partial_\nu u_w\|_{L^2(\Sigma_1)}. \tag{61}$$

**Proof of Theorem 14**

Note that $u = u_a - u_0$ is the solution of the **variationnal problem**

$$\begin{cases} \displaystyle\int_\Omega u''(t)vdx = \int_\Omega \nabla u(t)\cdot\nabla vdx - \int_{\Gamma_1} au'(t)v - \int_{\Gamma_1} au_0'(t)v, \quad v \in V. \\ u(0) = 0, \quad u'(0) = 0. \end{cases} \tag{62}$$

For $k,\ \ell \in \mathbb{Z}$, set

$$\lambda_{k\ell} = [(k+1/2)^2 + (\ell+1/2)^2]\pi^2$$
$$\phi_{k\ell}(x,y) = 2\cos((k+1/2)\pi x)\cos((\ell+1/2)\pi y).$$

Fix $k$ and $\ell$.Let $\lambda(t) = \cos(\sqrt{\lambda_{k\ell}}t)$ and define $w_a \in V'$ by

$$w_a(v) = -\sqrt{\lambda_{k\ell}}\int_{\Gamma_1} a\phi_{k\ell}v.$$

Then (62) becomes

$$\begin{cases} \displaystyle\int_\Omega u''(t)vdx = \int_\Omega \nabla u(t)\cdot\nabla vdx - \int_{\Gamma_1} au'(t)v + \lambda(t)w_a(v), \quad v \in V. \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

We get by applying Proposition 3

$$\|w_a\|_{V'} \leq Ce^{\lambda_{k\ell}\tau^2}\|\partial_\nu u\|_{L^2(\Sigma_1)}. \tag{63}$$

But

$$\begin{aligned} a_1(0)\left|\int_{\Gamma_1}(a\phi_{k\ell})^2 d\sigma\right| &= \frac{1}{\sqrt{\lambda_{k\ell}}}\left|w_a((a_1\otimes a_2)\phi_{k\ell})\right| \\ &\leq \frac{1}{\sqrt{\lambda_{k\ell}}}\|w_a\|_{V'}\|(a_1\otimes a_2)\phi_{k\ell}\|_V, \end{aligned} \tag{64}$$

where we used $a_1(0) = a_2(0)$.

Also

$$\|(a_1\otimes a_2)\phi_{k\ell}\|_V \leq C_0\sqrt{\lambda_{kl}}\|a_1\otimes a_2\|_{H^1(\Omega)}, \tag{65}$$

Here $C_0$ is a constant independent on $a$ and $\phi_{k\ell}$.

A combination of (63), (64) and (65) yields

$$a_1(0)\left(\|a_1\phi_k\|^2_{L^2((0,1))}+\|a_2\phi_\ell\|^2_{L^2((0,1))}\right) \\ \leq C\|a_1\|_{H^1(0,1)}\|a_2\|_{H^1(0,1)}e^{\lambda_{k\ell}\tau^2/2}\|\partial_\nu u\|_{L^2(\Sigma_1)},$$

where $\phi_k(s)=\sqrt{2}\cos((k+1/2)\pi s)$. This and the fact that $m\leq a_j(0)$ and $\|a_j\|_{H^1((0,1))}\leq M$ imply

$$\|a_1\phi_k\|^2_{L^2((0,1))}+\|a_2\phi_\ell\|^2_{L^2((0,1))}\leq C\frac{M^2}{m}e^{\lambda_{k\ell}\tau^2/2}\|\partial_\nu u\|_{L^2(\Sigma_1)},$$

Whence, where $j=1$ or $2$,

$$\|a_j\phi_k\|^2_{L^2((0,1))}\leq C\frac{M^2}{m}e^{k^2\tau^2\pi^2}\|\partial_\nu u\|_{L^2(\Sigma_1)}.$$

The rest of the proof is quite similar to the preceding ones.

## Final comments

- Section 1 was taken from

**M. Choulli**, Various stability estimates for the problem of determining an initial heat distribution from a single measurement, **arXiv :1512 :07421**.

- Sections 2, 3 and 5 was prepared from

**K. Ammari** and **M. Choulli**, Logarithmic stability in determining two coefficients in a dissipative wave equation. Extensions to clamped Euler-Bernoulli beam and heat equations, **J. Diff. Equat.** 259 (7) (2015) 3344-3365.

The results of section 4 suggest that the result in Section 5 can be improved to Hölder stability.

- Section 4 is taken from

**K. Ammari**, **M. Choulli** and **F. Triki**, Hölder stability in determining the potential and the damping coefficient in a wave equation, **arXiv :1609.06102**.

• The results presented in Sections 8 and 9 were published in

**K. Ammari**, **M. Choulli** and **F. Triki**, Determining the potential in a wave equation without a geometric condition. Extension to the heat equation **Proc. Amer. Math. Soc.** 144 (10) (2016) 4381-4392.

• The result of Section 8 is the first attempt to solve the stability issue of the problem of determining the damping boundary coefficient in a wave equation from the initial-to-boundary operator :

**K. Ammari** and **M. Choulli**, Logarithmic stability in determining a boundary coefficient in an ibvp for the wave equation, to appear in **Dynamics of PDE**.

The problem in all of its generality remains open.