

Controllability of non-scalar parabolic systems: Some recent results and phenomena

Manuel González-Burgos,
UNIVERSIDAD DE SEVILLA

Marrakesh Workshop on Control, Inverse Problems and Stabilization of Infinite Dimensional Systems

Marrakesh, December 2016

General objective:

Study some null controllability problems for **non-scalar parabolic systems**.

Non-scalar parabolic systems: arise in chemical reactions, when we model problems from the Biology and in a wide variety of physical situations.

In this course we will deal with non-scalar systems which in fact are **coupled parabolic scalar equations**. We do not present results relating to the controllability problems of systems which come from fluid mechanics as Stokes, Navier-Stokes, ...

GOAL:

- 1 Show the important differences between scalar and non-scalar problems.
- 2 Give necessary and sufficient conditions (**Kalman conditions**) which characterize the controllability properties of these systems.
- 3 Show some hyperbolic phenomena related to the controllability properties of these systems.

We will only deal with

- 1 **Linear systems**
- 2 In general, “simple” Parabolic Systems.

- 1 Introduction
- 2 The parabolic scalar case
 - The one-dimensional case: The moment method
 - General case: Carleman Inequalities
 - Final comments in the scalar case
- 3 Finite-dimensional systems
- 4 Distributed controllability of 2×2 linear systems
- 5 Boundary controllability of a 2×2 linear system
- 6 A generalization: Cascade systems
- 7 The Kalman condition for a class of parabolic systems. Distributed controls
- 8 The Kalman condition for a class of parabolic systems. Boundary controls
- 9 New phenomena: Minimal time of controllability
- 10 New phenomena: Dependence on the position of the control set
- 11 Further results
- 12 Comments and open problems

1. Introduction

1. Introduction

Let us fix $T > 0$ and let H and U be two separable Hilbert spaces. Let us consider the autonomous system:

$$(1) \quad \begin{cases} y' = Ay + Bu & \text{on } (0, T), \\ y(0) = y_0 \in H. \end{cases}$$

A and B are “appropriate” operators, $y_0 \in H$ is **the initial datum** at $t = 0$ and $u \in L^2(0, T; U)$ is the **control** (exerted by means of the operator B).

Assume the problem is well-posed: $\forall (y_0, u)$ there exists a unique weak solution $y \in C^0([0, T]; H)$ to (1) which depends continuously on the data.

Let us denote by $y(t; y_0, u) \in H$ the solution to the system at time $t \in [0, T]$.

Example

$H = \mathbb{R}^n$ ($n \geq 1$), $U = \mathbb{R}^m$ ($m \geq 1$), $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$: ordinary differential system with n variables and m controls.

1. Introduction

- **Exact Controllability:** System (1) is **exactly controllable** at time T if $\forall (y_0, y_1) \in H \times H$, there exists $u \in L^2(0, T; U)$ s.t. the solution y of (1) satisfies $y(T; y_0, u) = y_1$.
- **Controllability to trajectories:** System (1) is **controllable to trajectories** at time T if $\forall (y_0, \hat{y}_0) \in H \times H$ and $\hat{u} \in L^2(0, T; U)$, there exists $u \in L^2(0, T; U)$ s.t. the corresponding weak solution to (1) satisfies $y(T; y_0, u) = y(T; \hat{y}_0, \hat{u})$.
- **Null Controllability:** System (1) is **null controllable** at time T if $\forall y_0 \in H$ there exists $u \in L^2(0, T; U)$ s.t. $y(T; y_0, u) = 0$.
Linear case: Controllability to trajectories and null controllability are equivalent.
- **Approximate Controllability:** System (1) is **approximately controllable** at time T if $\forall (y_0, y_1) \in H \times H$, and every $\varepsilon > 0$, there exists $u \in L^2(0, T; U)$ s.t.

$$\|y(T; y_0, u) - y_1\|_H \leq \varepsilon.$$

2. The parabolic scalar case

Remark

Problem (1) is **linear**. Then, System (1) is **null controllable** at time T **if and only if** the system is **exactly controllable to the trajectories** at time T . ■

Remark

We will deal with parabolic problems. So, due to the **regularizing effect** of these problems, it is well-known that the exact controllability result fails.

Therefore, in this course we will study **null or approximate controllability** results for the system under consideration.

2. The parabolic scalar case

2. The parabolic scalar case

In this course we are going to deal with **time-dependent second order elliptic** operators. Thus, let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2 and let us fix $T > 0$.

Notation: $Q_T = \Omega \times (0, T)$, $\Sigma_T = \partial\Omega \times (0, T)$ and, for $\mathcal{O} \subseteq \Omega$ or $\mathcal{O} \subseteq \partial\Omega$, $1_{\mathcal{O}}$ denotes the characteristic function of the set \mathcal{O} .

Let $L(t)$ be the operator given by:

$$(2) \quad L(t)y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x, t) \frac{\partial y}{\partial x_j} \right) + D(x, t) \cdot \nabla y + c(x, t)y.$$

The coefficients of L satisfy

$$(3) \quad \begin{cases} \alpha_{ij} \in W^{1,\infty}(Q_T) \quad (1 \leq i, j \leq N), \quad D \in L^\infty(Q_T; \mathbb{R}^N), \quad c \in L^\infty(Q_T), \\ \alpha_{ij}(x, t) = \alpha_{ji}(x, t) \quad \forall (x, t) \in Q_T, \end{cases}$$

and the **uniform elliptic condition**: there exists $a_0 > 0$ such that

$$(4) \quad \sum_{i,j=1}^N \alpha_{ij}(x, t) \xi_i \xi_j \geq a_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \forall (x, t) \in Q_T.$$

2. The parabolic scalar case

Let $\omega \subseteq \Omega$ be an open subset, $\Gamma_0 \subseteq \partial\Omega$ a relative open subset and let us fix $T > 0$.

We consider the **linear** problems for the **operator** $L(t)$:

$$(5) \quad \begin{cases} \partial_t y + L(t)y = v1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

$$(6) \quad \begin{cases} \partial_t y + L(t)y = 0 & \text{in } Q_T, \\ y = h1_{\Gamma_0} \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

In (5) and (6), $y(x, t)$ is the state, y_0 is the **initial datum** and v and h are the control functions (which are localized in ω -**distributed control**- or on Γ_0 -**boundary control**-).

2. The parabolic scalar case

Let $\omega \subseteq \Omega$ be an open subset, $\Gamma_0 \subseteq \partial\Omega$ a relative open subset and let us fix $T > 0$.

We consider the **linear** problems for the **operator** $L(t)$:

$$(5) \quad \begin{cases} \partial_t y + L(t)y = v1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

$$(6) \quad \begin{cases} \partial_t y + L(t)y = 0 & \text{in } Q_T, \\ y = h1_{\Gamma_0} \text{ on } \Sigma, \quad y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

In (5) and (6), $y(x, t)$ is the state, y_0 is the **initial datum** and v and h are the control functions (which are localized in ω -**distributed control**- or on Γ_0 -**boundary control**-).

Question: Functional spaces for y_0 , v and h ?

2. The parabolic scalar case

CONTROL SPACES:

- **Distributed control problem:** We can take $L^2(Q_T)$ as **control space** and $L^2(\Omega)$ as **initial datum space**. The problem is **well-posed**: $\forall y_0 \in L^2(\Omega)$ and $v \in L^2(Q_T)$ there exists a unique weak solution to (5) $y \in C^0([0, T]; L^2(\Omega))$ which depends continuously on the data.

2. The parabolic scalar case

CONTROL SPACES:

- **Distributed control problem:** We can take $L^2(Q_T)$ as **control space** and $L^2(\Omega)$ as **initial datum space**. The problem is **well-posed**: $\forall y_0 \in L^2(\Omega)$ and $v \in L^2(Q_T)$ there exists a unique weak solution to (5) $y \in C^0([0, T]; L^2(\Omega))$ which depends continuously on the data.
- **Boundary control problem:**
 - 1 If in (2), $D \equiv 0$ in Q_T , we can take $L^2(\Sigma_T)$ as **control space** and $H^{-1}(\Omega)$ as **initial datum space**. Again, the problem is **well-posed**: $\forall y_0 \in H^{-1}(\Omega)$ and $h \in L^2(\Sigma_T)$ there exists a unique weak solution to (6) $y \in C^0([0, T]; H^{-1}(\Omega))$ which depends continuously on the data. **Solution defined by transposition.**

2. The parabolic scalar case

CONTROL SPACES:

- **Distributed control problem:** We can take $L^2(Q_T)$ as **control space** and $L^2(\Omega)$ as **initial datum space**. The problem is **well-posed**: $\forall y_0 \in L^2(\Omega)$ and $v \in L^2(Q_T)$ there exists a unique weak solution to (5) $y \in C^0([0, T]; L^2(\Omega))$ which depends continuously on the data.
- **Boundary control problem:**
 - 1 If in (2), $D \equiv 0$ in Q_T , we can take $L^2(\Sigma_T)$ as **control space** and $H^{-1}(\Omega)$ as **initial datum space**. Again, the problem is **well-posed**: $\forall y_0 \in H^{-1}(\Omega)$ and $h \in L^2(\Sigma_T)$ there exists a unique weak solution to (6) $y \in C^0([0, T]; H^{-1}(\Omega))$ which depends continuously on the data. **Solution defined by transposition.**
 - 2 In the general case, we can take $L^2(\Omega)$ as **initial datum space** and

$$X(\Gamma_0) = \{h : h = H|_{\Sigma_T} \text{ with } H \in L^2(0, T; H_0^1(\tilde{\Omega})), H_t \in L^2(0, T; H^{-1}(\tilde{\Omega}))\},$$

as **control space**, where $\tilde{\Omega}$ is an open set s.t. $\Omega \subset \tilde{\Omega}$, $\partial\Omega \cap \tilde{\Omega} \subset\subset \Gamma_0$ and $\tilde{\Omega} \setminus \bar{\Omega} \neq \emptyset$. The problem is **well-posed** and the solution depends continuously on the data.

2. The parabolic scalar case

Theorem

Let us fix $T > 0$. The following conditions are equivalent

- 1 For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\omega \subset \Omega$, nonempty open subset, and any coefficients α_{ij} ($1 \leq i, j \leq N$), D and c , satisfying (3) and (4), System (5) is null controllable in $L^2(\Omega)$ at time $T > 0$ with **distributed controls** $v \in L^2(Q_T)$.
- 2 For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\Gamma_0 \subset \partial\Omega$, nonempty relative open subset, and any coefficients α_{ij} ($1 \leq i, j \leq N$), D and c , satisfying (3) and (4), System (6) is null controllable in $L^2(\Omega)$ at time $T > 0$ with **boundary controls** $h \in L^2(0, T; H^{1/2}(\partial\Omega))$.

2. The parabolic scalar case

Theorem

Let us fix $T > 0$. The following conditions are equivalent

- 1 For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\omega \subset \Omega$, nonempty open subset, and any coefficients α_{ij} ($1 \leq i, j \leq N$), D and c , satisfying (3) and (4), System (5) is null controllable in $L^2(\Omega)$ at time $T > 0$ with **distributed controls** $v \in L^2(Q_T)$.
- 2 For any $\Omega \subset \mathbb{R}^N$, bounded open set with Ω having a C^2 boundary, any $\Gamma_0 \subset \partial\Omega$, nonempty relative open subset, and any coefficients α_{ij} ($1 \leq i, j \leq N$), D and c , satisfying (3) and (4), System (6) is null controllable in $L^2(\Omega)$ at time $T > 0$ with **boundary controls** $h \in L^2(0, T; H^{1/2}(\partial\Omega))$.

Proof: We will use in a **fundamental way** that the problem under consideration is **scalar** (in fact, same number of equations and controls). We follow some ideas from [BODART, G.-B., PÉREZ-GARCÍA] Comm. PDE (2004) and [G.-B., PÉREZ-GARCÍA] Asymp. Anal. (2006). ...

2. The parabolic scalar case

Remark (Regularizing effect)

The previous proof shows that if the distributed and boundary null controllability results for Systems (5) and (6) are valid with controls in $L^2(Q_T)$ and $L^2(0, T; H^{1/2}(\partial\Omega))$, then the previous systems are null controllable with controls in $L^\infty(Q_T)$ and $L^\infty(\Sigma_T)$ (and even better for regular coefficients). ■

Remark

In the proof of Theorem 1 we have strongly used that the operator $\partial_t + L(t)$ is **scalar**. We will see that the previous equivalence is not valid for **non-scalar parabolic operators**. ■

2. The parabolic scalar case

From now on, we will concentrate on the **distributed control** problem (5).

Let us introduce the **adjoint problem**

$$(7) \quad \begin{cases} -\partial_t \varphi + L^*(t)\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_T & \text{in } \Omega, \end{cases}$$

where $\varphi_T \in L^2(\Omega)$ is given and $L^*(t)$ is the operator given by

$$L^*(t)\varphi = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x, t) \frac{\partial \varphi}{\partial x_j} \right) - \nabla \cdot (D\varphi) + c(x, t)\varphi \text{ a.e. in } Q_T.$$

This problem is also well-posed and the solution depends continuously on φ_T : there exists a constant $\tilde{C} > 0$ such that $\forall \varphi_T \in L^2(\Omega)$ System (7) has only **one solution** $\varphi \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ and it satisfies

$$\|\varphi\|_{L^2(0, T; H_0^1(\Omega))} + \|\varphi\|_{C^0([0, T]; L^2(\Omega))} \leq \tilde{C} \|\varphi_T\|_{L^2(\Omega)}.$$

2. The parabolic scalar case

Theorem (Observability Inequality)

Under the previous assumptions, System (5) is null controllable at time $T > 0$ if and only if there exists a constant $C_T > 0$ s.t.

$$(8) \quad \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C_T \iint_{\omega \times (0, T)} |\varphi|^2 dx dt, \quad \forall \varphi_T \in L^2(\Omega),$$

where φ is the solution of (7) associated to φ_T .

2. The parabolic scalar case

Theorem (Observability Inequality)

Under the previous assumptions, System (5) is null controllable at time $T > 0$ if and only if there exists a constant $C_T > 0$ s.t.

$$(8) \quad \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C_T \iint_{\omega \times (0, T)} |\varphi|^2 dx dt, \quad \forall \varphi_T \in L^2(\Omega),$$

where φ is the solution of (7) associated to φ_T .

Remark

The **Observability Inequality** (8) in particular implies a better result: If (8) holds then, $\forall y_0 \in L^2(\Omega)$ there is a distributed control $v \in L^2(Q_T)$ s.t.

$$\|v\|_{L^2(Q_T)}^2 \leq C_T \|y_0\|_{L^2(\Omega)}^2 \quad \text{and} \quad y(\cdot, T) = 0,$$

being y the solution to (5) corresponding to y_0 and $C_T > 0$ the constant in (8).

2. The parabolic scalar case

Remark (Control cost)

The previous remark and inequality (8) provide an estimate of **the cost of the control** for system (5): If (8) holds at time $T > 0$, then

$$\mathcal{Z}_T(y_0) := \{v \in L^2(Q_T) : y(T; y_0, v) = 0\} \neq \emptyset, \quad \forall y_0 \in L^2(\Omega).$$

We can then define the control cost for system (5) at time T as

$$\mathcal{K}(T) = \sup_{\|y_0\|_{L^2(\Omega)}=1} \left(\inf_{v \in \mathcal{Z}_T(y_0)} \|v\|_{L^2(Q_T)} \right), \quad \forall T > 0.$$

Thus, $\mathcal{K}(T) \leq \sqrt{C_T}$. On the other hand, if $\mathcal{Z}_T(y_0) \neq \emptyset$, for any $y_0 \in L^2(\Omega)$, then, the observability inequality (8) for the adjoint system (7) holds with $C_T = \mathcal{K}(T)^2$. It is then clear that

$$\mathcal{K}(T) = \inf \left\{ \sqrt{C_T} : C_T > 0 \text{ is such that (8) holds} \right\}.$$

2. The parabolic scalar case

1. The one-dimensional case: The moment method

We follow [FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971).

2. The parabolic scalar case

1. The one-dimensional case: The moment method

Consider the **boundary null controllability problem** for the classical one-dimensional heat equation in $(0, \pi)$ (for simplicity):

$$(9) \quad \left\{ \begin{array}{ll} y_t - y_{xx} = 0 & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = v, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{array} \right.$$

with $y_0 \in H^{-1}(0, \pi)$ and $v \in L^2(0, T)$. The problem is **well-posed** and the solution (**defined by transposition**) depends continuously on the data y_0 and v . The operator $-\partial_{xx}$ on $(0, \pi)$ with homogenous Dirichlet boundary conditions admits a sequence of **eigenvalues** and **normalized eigenfunctions** given by

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi)$$

which is a Hilbert basis of $L^2(0, \pi)$. In the sequel, we will use the notation

$$y_k = (y, \phi_k)_{L^2(0, \pi)}, \quad \forall y \in L^2(0, \pi).$$

2. The parabolic scalar case

1. The one-dimensional case: The moment method

The idea of the **moment method** is simple: Given $y_0 \in H^{-1}(0, \pi)$, $\varphi_T \in H_0^1(0, \pi)$ and $\mathbf{v} \in L^2(0, T)$, then

$$\langle y(\cdot, T), \varphi_T \rangle - \langle y_0, \varphi(\cdot, 0) \rangle = \int_0^T \mathbf{v}(t) \varphi_x(0, t) dt.$$

where y is the solution to (9) and φ is the solution to the **adjoint problem**

$$\begin{cases} -\varphi_t - \varphi_{xx} = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0, 1\} \times (0, T), \quad \varphi(\cdot, T) = \varphi_T & \text{in } (0, \pi). \end{cases}$$

2. The parabolic scalar case

1. The one-dimensional case: The moment method

The idea of the **moment method** is simple: Given $y_0 \in H^{-1}(0, \pi)$, $\varphi_T \in H_0^1(0, \pi)$ and $\mathbf{v} \in L^2(0, T)$, then

$$\langle y(\cdot, T), \varphi_T \rangle - \langle y_0, \varphi(\cdot, 0) \rangle = \int_0^T \mathbf{v}(t) \varphi_x(0, t) dt.$$

where y is the solution to (9) and φ is the solution to the **adjoint problem**

$$\begin{cases} -\varphi_t - \varphi_{xx} = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0, 1\} \times (0, T), \quad \varphi(\cdot, T) = \varphi_T & \text{in } (0, \pi). \end{cases}$$

Property

$\mathbf{v} \in L^2(0, \pi)$ is a **null control** for system (9) (i.e., $\mathbf{v} \in L^2(0, T)$ is a control s.t. the solution y to (9) satisfies $y(\cdot, T) = 0$ in $(0, \pi)$) if and only if

$$-\langle y_0, \varphi(\cdot, 0) \rangle = \int_0^T \mathbf{v}(t) \varphi_x(0, t) dt, \quad \forall \varphi_T \in H_0^1(0, \pi).$$

2. The parabolic scalar case

1. The one-dimensional case: The moment method

Given $y_0 \in H^{-1}(0, \pi)$, there exists a control $\mathbf{v} \in L^2(0, T)$ such that the solution y to (9) satisfies $y(\cdot, T) = 0$ in $(0, \pi)$ if and only if there exists

$\mathbf{v} \in L^2(0, T)$ satisfying

$$-\langle y_0, e^{-\lambda_k T} \phi_k \rangle = \int_0^T \mathbf{v}(t) e^{-\lambda_k(T-t)} \phi_{k,x}(0) dt, \quad \forall k \geq 1,$$

i.e., if and only if $\mathbf{v} \in L^2(0, T)$ and

$$\int_0^T e^{-\lambda_k t} \mathbf{v}(T-t) dt = -\frac{1}{k} \sqrt{\frac{\pi}{2}} e^{-\lambda_k T} y_{0,k} \equiv \mathbf{c}_k \quad \forall k \geq 1.$$

This problem is called a **moment problem**.

2. The parabolic scalar case

1. The one-dimensional case: The moment method

Given $y_0 \in H^{-1}(0, \pi)$, there exists a control $\mathbf{v} \in L^2(0, T)$ such that the solution y to (9) satisfies $y(\cdot, T) = 0$ in $(0, \pi)$ if and only if there exists

$\mathbf{v} \in L^2(0, T)$ satisfying

$$-\langle y_0, e^{-\lambda_k T} \phi_k \rangle = \int_0^T \mathbf{v}(t) e^{-\lambda_k(T-t)} \phi_{k,x}(0) dt, \quad \forall k \geq 1,$$

i.e., if and only if $\mathbf{v} \in L^2(0, T)$ and

$$\int_0^T e^{-\lambda_k t} \mathbf{v}(T-t) dt = -\frac{1}{k} \sqrt{\frac{\pi}{2}} e^{-\lambda_k T} y_{0,k} \equiv \mathbf{c}_k \quad \forall k \geq 1.$$

This problem is called a **moment problem**. We have the following result:

2. The parabolic scalar case

1. The one-dimensional case: The moment method

Given $y_0 \in H^{-1}(0, \pi)$, there exists a control $\mathbf{v} \in L^2(0, T)$ such that the solution y to (9) satisfies $y(\cdot, T) = 0$ in $(0, \pi)$ if and only if there exists

$\mathbf{v} \in L^2(0, T)$ satisfying

$$-\langle y_0, e^{-\lambda_k T} \phi_k \rangle = \int_0^T \mathbf{v}(t) e^{-\lambda_k(T-t)} \phi_{k,x}(0) dt, \quad \forall k \geq 1,$$

i.e., if and only if $\mathbf{v} \in L^2(0, T)$ and

$$\int_0^T e^{-\lambda_k t} \mathbf{v}(T-t) dt = -\frac{1}{k} \sqrt{\frac{\pi}{2}} e^{-\lambda_k T} y_{0,k} \equiv \mathbf{c}_k \quad \forall k \geq 1.$$

This problem is called a **moment problem**. We have the following result:

Theorem

*For any $y_0 \in H^{-1}(0, \pi)$ and $T > 0$, there exists $\mathbf{v} \in L^2(0, T)$ solution to the previous **moment problem**. That is, \mathbf{v} is a **null control** for equation (9).*

2. The parabolic scalar case

1. The one-dimensional case: The moment method

Proof: Biorthogonal Families: ([FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971)). There exists a family $\{q_k\}_{k \geq 1} \subset L^2(0, T)$ satisfying

$$\textcircled{1} \int_0^T e^{-\lambda_k t} q_l(t) dt = \delta_{kl}, \quad \forall k, l \geq 1.$$

$$\textcircled{2} \forall \varepsilon > 0, \exists C(\varepsilon, T) > 0 \text{ s.t. } \|q_k\|_{L^2(0, T)} \leq C(\varepsilon, T) e^{\varepsilon \lambda_k}.$$

The control is obtained as a linear combination of $\{q_k\}_{k \geq 1}$, that is,

$$v(T-t) = \sum_{k \geq 1} c_k q_k(t) = -\sqrt{\frac{\pi}{2}} \sum_{k \geq 1} \frac{1}{k} e^{-\lambda_k T} y_{0,k} q_k(t)$$

and the previous bounds are used to prove that this combination converges in $L^2(0, T)$. ■

Two ingredients:

Existence and **bounds** of a biorthogonal family to real exponentials.

2. The parabolic scalar case

1. The one-dimensional case: The moment method

Remark

Theorem 2.2 is a consequence of the existence of a **biorthogonal family** in $L^2(0, T)$ to the sequence $\{e^{-\lambda_k t}\}_{k \geq 1}$ ($\lambda_k = k^2$), which satisfies appropriate **bounds**. In fact, in

- 1 LUXEMBURG, KOREVAAR, Trans. Amer. Math. Soc. 157 (1971),
- 2 FATTORINI, RUSSELL, Quart. Appl. Math. 32 (1974/75),
- 3 HANSEN, J. Math. Anal. Appl. 158 (1991), ...

it is proved a general result on existence of a **biorthogonal family** in $L^2(0, T)$ to $\{e^{-\Lambda_k t}\}_{k \geq 1}$ which satisfies appropriate **bounds** for sequences $\Lambda = \{\Lambda_k\}_{k \geq 1} \subset \mathbb{R}_+$ such that

$$\sum_{k \geq 1} \frac{1}{\Lambda_k} < \infty \quad \text{and} \quad |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1.$$

for a constant $\rho > 0$. ■

2. The parabolic scalar case

1. The one-dimensional case: The moment method

Consequence:

The previous result is valid for any nonempty bounded interval (a, b) and for any second order operator **self-adjoint elliptic operator**

$$Ly = -(\alpha(x)y_x)_x + c(x)y,$$

with $\alpha \in C^1([a, b])$ and $\alpha > 0$ in (a, b) , and $c \in C^0([a, b])$. **Then**, if we apply Theorem 1, we also get a **distributed controllability** result for the problem

$$\begin{cases} y_t + Ly = v1_\omega & \text{in } Q_T = (a, b) \times (0, T), \\ y(a, \cdot) = 0, \quad y(b, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (a, b), \end{cases}$$

with $y_0 \in L^2(0, \pi)$ and $\omega \subseteq (a, b)$, a nonempty open subset.

2. The parabolic scalar case

2. General case: Carleman Inequalities

We follow [FURSIKOV,IMANUVILOV] 1996 and [IMANUVILOV,YAMAMOTO] 2003.

2. The parabolic scalar case

2. General case: Carleman Inequalities

We will consider the following parabolic equation:

$$(10) \quad \begin{cases} -\partial_t z + L_0(t)z = F_0 + \sum_{i=1}^N \frac{\partial F_i}{\partial x_i} & \text{in } Q_T, \\ z = 0 \text{ on } \Sigma_T, \quad z(\cdot, T) = z_T & \text{in } \Omega, \end{cases}$$

with $z_T \in L^2(\Omega)$, $F_i \in L^2(Q_T)$, $i = 0, 1, \dots, N$, and $L_0(t)$ the self-adjoint parabolic operator given by

$$L_0(t)y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x, t) \frac{\partial y}{\partial x_j} \right)$$

with coefficients α_{ij} satisfying (3) (**regularity**) and (4) (**uniform elliptic condition**).

2. The parabolic scalar case

2. General case: Carleman Inequalities

Lemma

Let $\mathcal{B} \subset \Omega$ be a nonempty open subset and $d \in \mathbb{R}$. Then, $\exists \beta_0 \in C^2(\bar{\Omega})$ (**positive** and only depending on Ω and \mathcal{B}) and $\tilde{C}_0, \tilde{\sigma}_0 > 0$ (only depending on Ω , \mathcal{B} and d) s.t. for every $z_T \in L^2(\Omega)$, the solution z to (10) satisfies

$$(11) \quad \left\{ \begin{array}{l} \mathcal{I}(d, z) \leq \tilde{C}_0 \left(s^d \iint_{\mathcal{B} \times (0, T)} e^{-2s\beta} \gamma(t)^d |z|^2 \right. \\ \left. + s^{d-3} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-3} |F_0|^2 + s^{d-1} \sum_{i=1}^N \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-1} |F_i|^2 \right), \end{array} \right.$$

$$\forall s \geq \tilde{\sigma}_0 = \tilde{\sigma}_0 (T + T^2); \quad \boxed{\gamma(t) = t^{-1}(T-t)^{-1}}, \quad \boxed{\beta(x, t) = \beta_0(x)/t(T-t)}$$

$$\text{and } \boxed{\mathcal{I}(d, z) \equiv s^{d-2} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-2} |\nabla z|^2 + s^d \iint_{Q_T} e^{-2s\beta} \gamma(t)^d |z|^2}.$$

2. The parabolic scalar case

2. General case: Carleman Inequalities

Lemma

When $F_i \equiv 0$ for $1 \leq i \leq N$, $\exists \tilde{C}_1$ and $\tilde{\sigma}_1$ (which only depend on Ω , \mathcal{B} and d) s.t., $\forall z_T \in L^2(\Omega)$, the solution z to (10) satisfies

(12)

$$\mathcal{I}_1(d, z) \leq \tilde{C}_1 \left(s^d \iint_{\mathcal{B} \times (0, T)} e^{-2s\beta} \gamma(t)^d |z|^2 + s^{d-3} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-3} |F_0|^2 \right),$$

for all $s \geq \tilde{s}_1 = \tilde{\sigma}_1 (T + T^2)$ where

$$\mathcal{I}_1(d, z) \equiv s^{d-4} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-4} \left(|\partial_t z|^2 + \sum_{i,j=1}^N \left| \frac{\partial^2 z}{\partial x_i \partial x_j} \right|^2 \right) + \mathcal{I}(d, z).$$

Proof: See [FURSIKOV, IMANUVILOV] 1996; [IMANUVILOV, YAMAMOTO] (2003) and [FERNÁNDEZ-CARA, GUERRERO] SICON (2006).

2. The parabolic scalar case

2. General case: Carleman Inequalities

Recall that our objective is to prove a **null controllability** result at time T for

$$(5) \quad \boxed{\begin{cases} \partial_t y + L(t)y = v1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}}$$

with $L(t)$ given by:

$$\begin{cases} L(t)y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x,t) \frac{\partial y}{\partial x_j} \right) + D(x,t) \cdot \nabla y + c(x,t)y \\ = L_0(t)y + D(x,t) \cdot \nabla y + c(x,t)y, \end{cases}$$

with coefficients α_{ij} satisfying (3) and (4). We also know that this is equivalent to the **observability inequality** (8)

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C_T \iint_{\omega \times (0,T)} |\varphi|^2 dxdt, \quad \forall \varphi_T \in L^2(\Omega),$$

for the solutions to the **adjoint problem** (7).

2. The parabolic scalar case

2. General case: Carleman Inequalities

Corollary

There exists a constant $C_0 = C_0(\Omega, \omega) > 0$ such that $\forall \varphi_T \in L^2(\Omega)$ and φ the corresponding solution to (7), the **observability inequality** (8) holds with

$$C_T = \exp \left(C_0 \left(1 + \frac{1}{T} + \|c\|_\infty^{2/3} + T\|c\|_\infty + (1 + T)\|D\|_\infty^2 \right) \right).$$

2. The parabolic scalar case

2. General case: Carleman Inequalities

Corollary

There exists a constant $C_0 = C_0(\Omega, \omega) > 0$ such that $\forall \varphi_T \in L^2(\Omega)$ and φ the corresponding solution to (7), the **observability inequality** (8) holds with

$$C_T = \exp \left(C_0 \left(1 + \frac{1}{T} + \|c\|_\infty^{2/3} + T\|c\|_\infty + (1+T)\|D\|_\infty^2 \right) \right).$$

Proof: We follow [FERNÁNDEZ-CARA,ZUAZUA] Ann. IHP (2000) and [DOUBOVA,FERNÁNDEZ-CARA,MG-B,ZUAZUA] SICON (2002).

The Carleman inequality (11) applied to problem (7) implies ($\mathcal{B} \equiv \omega$, $d = 3$ and $-\partial_t \varphi + L_0(t)\varphi = \nabla \cdot (D\varphi) - c(x, t)\varphi$) that $\forall s \geq \tilde{s}_0 = \tilde{\sigma}_0 (T + T^2)$:

$$\begin{aligned} & s \iint_{Q_T} e^{-2s\beta} \gamma(t) |\nabla \varphi|^2 + s^3 \iint_{Q_T} e^{-2s\beta} \gamma(t)^3 |\varphi|^2 \\ & \leq \tilde{C}_0 \left(s^3 \iint_{\omega \times (0, T)} e^{-2s\beta} \gamma(t)^3 |\varphi|^2 \right. \\ & \left. + \|c\|_\infty^2 \iint_{Q_T} e^{-2s\beta} |\varphi|^2 + s^2 \|D\|_\infty^2 \iint_{Q_T} e^{-2s\beta} \gamma(t)^2 |\varphi|^2 \right). \end{aligned}$$

2. The parabolic scalar case

2. General case: Carleman Inequalities

As a consequence we can prove that for

$s \geq C_1(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|D\|_\infty^2))$ ($C_1 = C_1(\Omega, \omega)$) one has

$$[s\gamma(t)]^3 - \tilde{C}_0\|c\|_\infty^2 - \tilde{C}_0[s\gamma(t)]^2\|D\|_\infty^2 \geq \frac{1}{2}[s\gamma(t)]^3.$$

Consequently, for $s = C_1(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|D\|_\infty^2))$ that

$$\iint_{Q_T} e^{-2s\beta} t^{-3} (T-t)^{-3} |\varphi|^2 \leq \tilde{C}_1 \iint_{\omega \times (0, T)} e^{-2s\beta} t^{-3} (T-t)^{-3} |\varphi|^2$$

and therefore

$$\iint_{\Omega \times (T/4, 3T/4)} |\varphi|^2 \leq e^{C(1+1/T+\|c\|_\infty^{2/3}+\|D\|_\infty^2)} \iint_{\omega \times (0, T)} |\varphi|^2.$$

This last inequality combined with **energy estimates** ($C = C(a_0) > 0$)

$$\frac{d}{dt} \left(e^{C(\|c\|_\infty + \|D\|_\infty^2)t} \int_{\Omega} |\varphi|^2(\cdot, t) \right) \geq 0 \quad \forall t \in [0, T]$$

implies (8) and the proof is complete. ■

2. The parabolic scalar case

2. General case: Carleman Inequalities

Corollary

Let us fix $T > 0$, $\Omega \subset \mathbb{R}^N$, $\omega \subseteq \Omega$ and $\Gamma_0 \subseteq \partial\Omega$ (arbitrary) as before. Then, there exist positive constants $C_0 = C_0(\Omega, \omega)$ and $\widehat{C}_0 = \widehat{C}_0(\Omega, \Gamma_0)$ s.t.

- ① $\forall y_0 \in L^2(\Omega)$ there is a control $v \in L^2(\Omega)$ which satisfies

$$\|v\|_{L^2(Q_T)}^2 \leq e^{C_0(1+1/T+\|c\|_\infty^{2/3}+T\|c\|_\infty+(1+T)\|D\|_\infty^2)} \|y_0\|_{L^2(\Omega)}^2,$$

and $y(\cdot, T) = 0$ in Ω , (y is the solution to (5) associated to y_0 and v).

- ② $\forall y_0 \in L^2(\Omega)$ there is a control $h \in L^2(0, T; H^{1/2}(\Omega))$ which satisfies

$$\|h\|_{L^2(0, T; H^{1/2}(\Omega))}^2 \leq e^{\widehat{C}_0(1+1/T+\|c\|_\infty^{2/3}+T\|c\|_\infty+(1+T)\|D\|_\infty^2)} \|y_0\|_{L^2(\Omega)}^2,$$

and $y(\cdot, T) = 0$ in Ω , (y is the solution to (6) associated to y_0 and v and, in fact, $y \in L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$).

2. The parabolic scalar case

2. General case: Carleman Inequalities

Remark

It is important to point out that the **boundary null controllability** result for problem (6), when the coefficient D of $L(t)$ (see (2)) is regular enough, can be obtained from an appropriate boundary Carleman inequality for problem (10) with $F_i \equiv 0$, $1 \leq i \leq N$. This Carleman inequality is like (12) for an appropriate weight function $\tilde{\beta}_0 \in C^2(\bar{\Omega})$ (which depends only on Ω and Γ_0) instead of β_0 and with the local term

$$s^{d-2} \iint_{\Gamma_0 \times (0, T)} e^{-2s \frac{\tilde{\beta}_0}{t(T-t)}} \gamma(t)^{d-2} \left| \frac{\partial z}{\partial n} \right|^2$$

instead of the integral over $\mathcal{B} \times (0, T)$ in the right hand side of (12) (z is the solution to (10) associated to $z_T \in L^2(\Omega)$). ■

2. The parabolic scalar case

3. Final comments in the scalar case

2. The parabolic scalar case

3. Final comments in the scalar case

1. The null controllability property for the N -dimensional case was solved independently by G. Lebeau and L. Robbiano (for the heat equation) and by A. Fursikov and O. Imanuvilov (for a general parabolic equation). With a different approach, Lebeau-Robbiano obtained the distributed null controllability result for System (5)

$$\begin{cases} \partial_t y + L_0 y = v 1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

when L_0 is a self-adjoint elliptic operator independent of t . For more details, see [LEBEAU,ROBBIANO] Comm. P.D.E. (1995).

2. Until now, we have only dealt with the **null controllability** problem for a scalar parabolic system with distributed and boundary controls. For the corresponding **approximate controllability** we can obtain similar results:

2. The parabolic scalar case

3. Final comments in the scalar case

Approximate controllability

Proposition (Distributed control)

System (5) is **approximately controllable** at time $T > 0$ if and only if the **adjoint problem** (7) satisfies the **unique continuation property**: “If φ is a solution to (7) and $\varphi = 0$ in $\omega \times (0, T)$, then $\varphi \equiv 0$ in Q_T ”.

Remark (Boundary control)

In the case of System (6) we can get a similar result. In this case the **unique continuation property** for System (7) is: “If φ is a solution to (7) and $\partial_n \varphi = 0$ on $\Gamma_0 \times (0, T)$, then $\varphi \equiv 0$ in Q_T ”.

Theorem

System (5) (resp. System (6)) is **approximately controllable** at time $T > 0$, for any ω and $T > 0$ (resp., for any Γ_0 and T).

2. The parabolic scalar case

3. Final comments in the scalar case

Remark

The **distributed controllability** result for System (5) **is equivalent** to the **boundary controllability** result for System (6).

Summarizing:

- System (5) and system (6) are approximately controllable and exactly controllable to trajectories at any time $T > 0$ for every geometrical data ω or Γ_0 .
- The controllability properties of both systems are equivalent. ■

2. The parabolic scalar case

3. Final comments in the scalar case

SOME REFERENCES

- 1 H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.
- 2 G. LEBEAU, L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. P.D.E. 20 (1995), no. 1-2, 335–356.
- 3 O. YU. IMANUVILOV, *Controllability of parabolic equations*, (Russian) Sb. Math. 186 (1995), no. 6, 879–900.
- 4 A. FURSIKOV, O. YU. IMANUVILOV, *Controllability of Evolution Equations*, Lecture Notes Series 34, Seoul National Univ., Seoul, 1996.
- 5 O. YU. IMANUVILOV, M. YAMAMOTO, *Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations*, Publ. Res. Inst. Math. Sci. **39** (2003), no. 2, 227–274.

3. Finite-dimensional systems

3. Finite-dimensional systems

Let us consider the **autonomous linear system**

$$(13) \quad y' = Ay + Bu \quad \text{on } [0, T], \quad y(0) = y_0,$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ are constant matrices, $y_0 \in \mathbb{C}^n$ and $u \in L^2(0, T; \mathbb{C}^m)$ is the control.

Problem:

Given $y_0, y_d \in \mathbb{C}^n$, is there a control $u \in L^2(0, T; \mathbb{C}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

3. Finite-dimensional systems

Let us consider the **autonomous linear system**

$$(13) \quad y' = Ay + Bu \quad \text{on } [0, T], \quad y(0) = y_0,$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ are constant matrices, $y_0 \in \mathbb{C}^n$ and $u \in L^2(0, T; \mathbb{C}^m)$ is the control.

Problem:

Given $y_0, y_d \in \mathbb{C}^n$, is there a control $u \in L^2(0, T; \mathbb{C}^m)$ such that the solution y to the problem satisfies

$$y(T) = y_d????$$

Let us define (*controllability matrix*)

$$[A \mid B] = (B, AB, A^2B, \dots, A^{n-1}B) \in \mathcal{L}(\mathbb{C}^{nm}; \mathbb{C}^n).$$

On the other hand, let $\{\theta_l\}_{1 \leq l \leq \hat{p}} \subset \mathbb{C}$ be the set of distinct eigenvalues of A^* . For $l : 1 \leq l \leq \hat{p}$, we denote by m_l the geometric multiplicity of θ_l . The sequence $\{w_{l,j}\}_{1 \leq j \leq m_l}$ will denote a basis of the eigenspace associated to θ_l .

3. Finite-dimensional systems

The following classical result can be found in

R. KALMAN, Y.-CH. HO, K. NARENDRA, *Controllability of linear dynamical systems*, 1963.

and gives a complete answer to the problem of controllability of finite dimensional autonomous linear systems:

Theorem

Under the previous assumptions, the following conditions are equivalent

- 1 System (13) is *exactly controllable* at time T , for every $T > 0$.
- 2 There exists $T > 0$ such that system (13) is *exactly controllable* at time T .
- 3 $\text{rank}[A \mid B] = n$ or $\ker[A \mid B]^* = \{0\}$ (*Kalman rank condition*).
- 4 *Hautus test*: $\text{rank} \begin{pmatrix} A^* - \theta_l I_n \\ B^* \end{pmatrix} = n, \quad \forall l : 1 \leq l \leq \hat{p}$.
- 5 $\text{rank}[B^* w_{l,1}, B^* w_{l,2}, \dots, B^* w_{l,m_l}] = m_l$, for every $l : 1 \leq l \leq \hat{p}$. ■

3. Finite-dimensional systems

Remark

- 1 The four controllability concepts (**exact**, **exact to trajectories**, **null** and **approximate controllability**) for System (13) are equivalent (**finite-dimensional space**).
- 2 Observe that $\{B^*w_{l,1}, B^*w_{l,2}, \dots, B^*w_{l,m_l}\} \subset \mathbb{C}^m$. Condition 5 in Theorem 4 says this set is linearly independent for any $l : 1 \leq l \leq \hat{p}$. In particular, $m_l \leq m \quad \forall l : 1 \leq l \leq \hat{p}$.
- 3 Given the o.d.s. (**adjoint problem**)

$$-\varphi' = A^*\varphi \quad \text{in } [0, T], \quad \varphi(T) = \varphi_T \in \mathbb{C}^n,$$

it is not difficult to prove the following result: “System (13) is **exactly controllable** at time T **if and only if** the following property for the **adjoint problem** holds (**unique continuation property**)

$$\text{If } B^*\varphi(\cdot) = 0 \text{ on } [0, T], \text{ then } \varphi_T \equiv 0.”$$

3. Finite-dimensional systems

Goal

We have a complete characterization of the controllability results for finite-dimensional linear ordinary differential systems (a **Kalman condition**). Is it possible to obtain similar results for Partial Differentials Systems? We will focus on coupled linear **parabolic** systems.

What are the possible generalizations to Systems of Parabolic Equations?

4. Distributed controllability of 2×2 linear systems

4. Distributed controllability of 2×2 linear systems

Let us consider the 2×2 linear reaction-diffusion system ($Q_T = \Omega \times (0, T)$)

$$(14) \quad \begin{cases} \partial_t y_1 + L_0^1(t)y_1 + a_{11}y_1 + a_{12}y_2 = v1_\omega & \text{in } Q_T, \\ \partial_t y_2 + L_0^2(t)y_2 + a_{21}y_1 + a_{22}y_2 = 0 & \text{in } Q_T, \\ y_i = 0 \text{ on } \Sigma_T = \partial\Omega \times (0, T), \quad y_i(\cdot, 0) = y_0^i \text{ in } \Omega, \quad 1 \leq i \leq 2, \end{cases}$$

where Ω , ω and T are as before, $a_{ij} = a_{ij}(x, t) \in L^\infty(Q_T)$ ($1 \leq i, j \leq 2$), $y_0^i \in L^2(\Omega)$ ($1 \leq i \leq 2$) and $L_0^k(t)$ is, for every $1 \leq k \leq 2$, the second order

operator $L_0^k(t)y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}^k(x, t) \frac{\partial y}{\partial x_j} \right)$ where α_{ij}^k satisfy (3) and (4).

Remark

System (14) is controlled by means of a scalar distributed control exerted on the right-hand side of the first equation. The second equation is indirectly controlled by the coupling term $a_{21}y_1$. **Necessary condition** $a_{21} \neq 0$ ($a_{21} \in L^\infty(Q_T)$).

4. Distributed controllability of 2×2 linear systems

Equivalently, the previous system can be written as

$$(15) \quad \begin{cases} \partial_t y + \widehat{L}(t)y + Ay = Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\widehat{L}(t)$ is the **matrix operator** given by $\widehat{L}(t) = \text{diag}(L_0^1(t), L_0^2(t))$, $y = (y_i)_{1 \leq i \leq 2}$ is the state and where

$$\begin{cases} y_0 = (y_0^i)_{1 \leq i \leq 2} \in L^2(\Omega; \mathbb{R}^n), & A(\cdot, \cdot) = (a_{ij}(\cdot, \cdot))_{1 \leq i, j \leq 2} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n)), \\ \text{and } B \equiv e_1 = (1, 0)^* \in \mathbb{R}^2 \end{cases}$$

are given. Let us observe that, for each $y_0 \in L^2(\Omega; \mathbb{R}^2)$ and $v \in L^2(Q_T)$, System (15) admits a **unique weak solution**

$$y \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^2)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^2)).$$

4. Distributed controllability of 2×2 linear systems

Assumption

We assume that the coupling coefficient $a_{21} \in L^\infty(Q_T)$ satisfies

$$(16) \quad \boxed{a_{21} \geq c_0 > 0} \text{ or } \boxed{-a_{21} \geq c_0 > 0} \text{ in } \omega_0 \times (0, T),$$

with $\omega_0 \subseteq \omega$ a new open subset.

As in the scalar case, the controllability result for system (15) is equivalent to the **observability inequality**: $\exists C_T > 0$ such that

$$\|\varphi_1(\cdot, 0)\|_{L^2}^2 + \|\varphi_2(\cdot, 0)\|_{L^2}^2 \leq C_T \iint_{\omega \times (0, T)} |\varphi_1(x, t)|^2 dx dt,$$

where φ is the solution associated to $\varphi_0 \in L^2(\Omega; \mathbb{R}^2)$ of the **adjoint problem**:

$$(17) \quad \begin{cases} -\varphi_t + \widehat{L}(t)\varphi + A^*\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

4. Distributed controllability of 2×2 linear systems

Theorem

Under assumption (16), there exist a positive function $\alpha_0 \in C^2(\bar{\Omega})$ (only depending on Ω and ω_0), two positive constants C_0 and σ_0 (only depending on Ω , ω_0 , c_0 , $\|a_{21}\|_\infty$ and d) such that, for every $\varphi_T \in L^2(Q_T; \mathbb{R}^2)$, the solution φ to the **adjoint problem** (17) satisfies

$$\mathcal{I}_1(d+3, \varphi_1) + \mathcal{I}_1(d, \varphi_2) \leq C_0 s^{d+4} \iint_{\omega \times (0, T)} e^{-2s\alpha\gamma(t)^{d+4}} |\varphi_1|^2,$$

$\forall s \geq s_0 = \sigma_0 \left[T + T^2 + T^2 \left(\|a_{11}\|_\infty^{2/3} + \|a_{12}\|_\infty^{1/3} + \|a_{22}\|_\infty^{2/3} \right) \right]$. In the previous inequality, $\gamma(t) = t^{-1}(T-t)^{-1}$, $\alpha(x, t) = \alpha_0(x)/t(T-t)$ and $\mathcal{I}_1(d, z)$ is given in Lemmas 2.3 and 2.4 (with α instead of β).

4. Distributed controllability of 2×2 linear systems

Proof: Given $\omega_0 \subset \omega$, we choose $\omega_1 \subset\subset \omega_0$. Let $\alpha_0 \in C^2(\bar{\Omega})$ be the function provided by Lemma 2.3 and associated to Ω and $\mathcal{B} \equiv \omega_1$. We will also consider $\alpha(x, t) = \alpha_0(x)/t(T-t)$ and $\gamma(t) = t^{-1}(T-t)^{-1}$. We will do the proof in two steps:

Step 1. Let φ be the solution to **adjoint system** associated to φ_T . Each component satisfies

$$-\partial_t \varphi_i + L_0^i(t) \varphi_i = -a_{1i} \varphi_1 - a_{2i} \varphi_2.$$

We begin applying inequality (12) with $\mathcal{B} = \omega_1$ to each function φ_i with $L_0 \equiv L_0^i$, $d = d + 3(2 - i)$ and the corresponding right-hand side:

$$\begin{aligned} \mathcal{I}_1(d+3, \varphi_1) &\leq \tilde{C}_1 \left(\iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+3} |\varphi_1|^2 \right. \\ &\quad \left. + \|a_{11}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_1|^2 + \|a_{21}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_2|^2 \right), \end{aligned}$$

4. Distributed controllability of 2×2 linear systems

$$\begin{aligned} \mathcal{I}_1(d+3, \varphi_1) &\leq \tilde{\mathcal{C}}_1 \left(\iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+3} |\varphi_1|^2 \right. \\ &\quad \left. + \|a_{11}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_1|^2 + \|a_{21}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_2|^2 \right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}_1(d, \varphi_2) &\leq \tilde{\mathcal{C}}_1 \left(\iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_2|^2 \right. \\ &\quad \left. + \|a_{12}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-3} |\varphi_1|^2 + \|a_{22}\|_\infty^2 \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-3} |\varphi_2|^2 \right). \end{aligned}$$

for all $s \geq \tilde{s}_1 = \tilde{\sigma}_1 (T + T^2)$.

4. Distributed controllability of 2×2 linear systems

Now if we take

$$s \geq s_1 = \sigma_1 \left[T + T^2 + T^2 \left(\|a_{11}\|_\infty^{2/3} + \|a_{12}\|_\infty^{1/3} + \|a_{22}\|_\infty^{2/3} \right) \right],$$

with $\sigma_1 = \sigma_1(\Omega, \omega_0, \|a_{21}\|_\infty) > 0$, we obtain the existence of a positive constants $C_1 = C_1(\Omega, \omega_0, \|a_{21}\|_\infty)$ such that if $s \geq s_1$, then

$$\mathcal{I}_1(d+3, \varphi_1) \leq C_1 \left(\iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+3} |\varphi_1|^2 + \mathcal{I}_1(d, \varphi_2) \right)$$

and

$$\mathcal{I}_1(d, \varphi_2) \leq C_1 \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_2|^2 + \frac{1}{4C_1} \mathcal{I}_1(d+3, \varphi_1).$$

4. Distributed controllability of 2×2 linear systems

From these two previous inequalities we can also get

$$\begin{aligned} \mathcal{I}_1(d+3, \varphi_1) + \mathcal{I}_1(d, \varphi_2) &\leq C_2 \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+3} |\varphi_1|^2 \\ &\quad + C_2 \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_2|^2, \end{aligned}$$

$\forall s \geq s_1$, with $C_2 = C_2(\Omega, \omega_0, \|a_{21}\|_\infty)$ a new positive constant.

Step 2. Thanks to the assumption (16):

$$(16) \quad \boxed{a_{21} \geq c_0 > 0} \text{ or } \boxed{-a_{21} \geq c_0 > 0} \text{ in } \omega_0 \times (0, T),$$

with $\omega_0 \subseteq \omega$ an open subset, and the cascade structure

$$a_{21}\varphi_2 = \partial_t \varphi_1 - L_0^1(t)\varphi_1 - a_{11}\varphi_1 \text{ in } Q_T,$$

can eliminate the second local terms. In order to carry this process out, we will need the following result:

4. Distributed controllability of 2×2 linear systems

Lemma

Let us assume (16). Then, given $\varepsilon > 0$, there exist a constant \tilde{C}_2 (only depending on Ω , c_0 and $\|a_{21}\|_\infty$), such that, if $s \geq s_1$, one has

$$\begin{aligned} \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_2|^2 &\leq \varepsilon \mathcal{I}_1(d, \varphi_2) \\ &+ \tilde{C}_2 \left(1 + \frac{1}{\varepsilon}\right) \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+4} |\varphi_1|^2. \end{aligned}$$

The proof of Theorem 4.1 is a consequence of this Lemma and the inequality

$$\begin{aligned} \mathcal{I}_1(d+3, \varphi_1) + \mathcal{I}_1(d, \varphi_2) &\leq C_2 \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+3} |\varphi_1|^2 \\ &+ C_2 \iint_{\omega_1 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^d |\varphi_2|^2. \end{aligned}$$

This ends the proof. ■

4. Distributed controllability of 2×2 linear systems

Summarizing

We have proved that the solutions to the **adjoint system**

$$(17) \quad \begin{cases} -\varphi_t + \widehat{L}(t)\varphi + A^*\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

satisfy the **Carleman inequality** $C_0 = C_0(\Omega, \omega_0, c_0, \|a_{21}\|_\infty, d)$

$$\mathcal{I}_1(d+3, \varphi_1) + \mathcal{I}_1(d, \varphi_2) \leq C_0 s^{d+4} \iint_{\omega \times (0, T)} e^{-2s\alpha\gamma(t)^{d+4}} |\varphi_1|^2,$$

$$\forall s \geq s_0 = \sigma_0 \left[T + T^2 + T^2 \left(\|a_{11}\|_\infty^{2/3} + \|a_{12}\|_\infty^{1/3} + \|a_{22}\|_\infty^{2/3} \right) \right].$$

($C_0 = C_0(\Omega, \omega_0, c_0, \|a_{21}\|_\infty, d)$ and $\sigma_0 = \sigma_0(\Omega, \omega_0, c_0, \|a_{21}\|_\infty, d)$ are positive constants).

4. Distributed controllability of 2×2 linear systems

As in the scalar case, combining the previous result and **energy inequalities** satisfied by the solutions of the **adjoint system** it is possible to prove an **observability inequality** for the **adjoint system** and deduce:

Corollary

Let us assume (16). Then, there exists a positive constant C (only depending on Ω , ω , c_0 and $\|a_{21}\|_\infty$) such that for every $y_0 \in L^2(\Omega; \mathbb{R}^2)$ there is a control $v \in L^2(\Omega)$ which satisfies

$$\|v\|_{L^2(Q_T)}^2 \leq e^{C\mathcal{H}} \|y_0\|_{L^2(\Omega; \mathbb{R}^2)}^2,$$

and $y(\cdot, T) = 0$ in Ω , with y the solution to (15) associated to y_0 and v . In the previous inequality, \mathcal{H} is given by

$$\mathcal{H} \equiv 1 + T + \frac{1}{T} + \|a_{11}\|_\infty^{2/3} + \|a_{12}\|_\infty^{1/3} + \|a_{22}\|_\infty^{2/3} + T \max_{1 \leq i, j \leq 2} \|a_{ij}\|_\infty.$$

4. Distributed controllability of 2×2 linear systems

Remark

- System (14) is always controllable if we exert a control in each equation (**two controls**).
- The controllability result for system (14) is **independent** of the operators $L_0^1(t)$ and $L_0^2(t)$. We will see that the situation is more intricate if in the system a general control vector $B \in \mathbb{R}^2$ is considered.
- The same result can be obtained for the distributed approximate controllability at time T . Therefore, **approximate** and **null controllability** are equivalent concepts (distributed case).
- Using a different technique (**fictitious controls**), it is possible to prove a null controllability result as in the previous corollary when the coupling matrix $A \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^2))$ satisfies: There exist an open subset $\omega_0 \subset\subset \omega$ and a positive constant a_0 s.t.

$$|a_{21}(x, t)| \geq a_0 > 0 \quad \text{in } \omega_0 \times (0, T). \quad \blacksquare$$

4. Distributed controllability of 2×2 linear systems

References

- 1 L. DE TERESA, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations 25 (2000), no. 1–2, 39–72.
- 2 F. AMMAR KHODJA, A. BENABDALLAH, C. DUPAIX ET I. KOSTIN, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim. 42 (2003), no. 5, 1661–1689.
- 3 M. G.-B., R. PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal. 46 (2006), no. 2, 123–162.
- 4 M. G.-B., L. DE TERESA, *Controllability results for cascade systems of m coupled parabolic PDEs by one control force*, Port. Math. 67 (2010), no. 1, 91–113.

5. Boundary controllability of a 2×2 linear system

5. Boundary controllability of a 2×2 linear system

Let us now consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$(18) \quad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q_T = (0, \pi) \times (0, T), \\ y|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{v}, \quad y|_{x=\pi} = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

with $y_0 \in H^{-1}(0, \pi; \mathbb{R}^2)$, $\mathbf{v} \in L^2(0, T)$ is the control and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad \boxed{(d_1 \neq d_2)}, \quad \text{and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Existence and uniqueness

For any $y_0 \in H^{-1}(0, \pi; \mathbb{R}^2)$ and $\mathbf{v} \in L^2(0, T)$, system (18) has a unique solution $y \in L^2(Q_T) \cap C^0([0, T]; H^{-1}(0, \pi; \mathbb{R}^2))$ defined by transposition.

5. Boundary controllability of a 2×2 linear system

Let us now consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$(18) \quad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q_T = (0, \pi) \times (0, T), \\ y|_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, \quad y|_{x=\pi} = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

with $y_0 \in H^{-1}(0, \pi; \mathbb{R}^2)$, $v \in L^2(0, T)$ is the control and

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad \boxed{(d_1 \neq d_2)}, \quad \text{and } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Question

Are the controllability properties of system (18) independent of d_1 and d_2 ???

NO.

5. Boundary controllability of a 2×2 linear system

As before, system (18) is **null controllable** at time T if and only if the **observability inequality**

$$\|\varphi_1(\cdot, 0)\|_{H_0^1(0, \pi)}^2 + \|\varphi_2(\cdot, 0)\|_{H_0^1(0, \pi)}^2 \leq C_T \int_0^T |\varphi_{1,x}(0, t)|^2 dt,$$

holds. Again φ is the solution associated to $\varphi_0 \in H_0^1(0, \pi; \mathbb{R}^2)$ of the **adjoint problem**:

$$(19) \quad \begin{cases} -\varphi_t - D\varphi_{xx} = A^*\varphi & \text{in } Q_T, \\ \varphi|_{x=0} = \varphi|_{x=\pi} = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi). \end{cases}$$

Let us see that, in general, this inequality fails (**even if** $a_{21} = 1 \neq 0$!!!!).

5. Boundary controllability of a 2×2 linear system

A necessary condition:

Proposition

Assume that system (18) is null controllable at time T ($d_1 \neq d_2$). Then $(\lambda_k = k^2)$,

$$d_1 \lambda_k \neq d_2 \lambda_j, \quad \forall k, j \geq 1 \quad (\iff \sqrt{d_1/d_2} \notin \mathbb{Q}).$$

Proof: By contradiction, assume that $d_1 \lambda_k = d_2 \lambda_j$ for some k, j and take $K = \max\{k, j\}$. The idea is transforming system (19) into an o.d.s. Recall that λ_k and ϕ_k are the eigenvalues and normalized eigenfunctions of $-\partial_{xx}$ on $(0, \pi)$ with homogenous Dirichlet boundary conditions:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

Idea: Take $\varphi_0 \in X_K = \{\varphi_0 = \sum_{\ell=1}^K a_\ell \phi_\ell : a_\ell \in \mathbb{R}^2\} \subset H_0^1(0, \pi; \mathbb{R}^2)$.

5. Boundary controllability of a 2×2 linear system

Consider also

$$B_K = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathbb{R}^{2K}, \quad (B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \quad \text{and}$$

$$\mathcal{L}_K^* = \text{diag} (-\lambda_1 D + A^*, -\lambda_2 D + A^*, \dots, -\lambda_K D + A^*) \in \mathcal{L}(\mathbb{R}^{2K}).$$

Taking in (19) arbitrary initial data $\varphi_{0,K} = \sum_{\ell=1}^K a_\ell \phi_\ell \in H_0^1(0, \pi; \mathbb{R}^2)$ where $a_\ell \in \mathbb{R}^2$, it is not difficult to see that system (19) is equivalent to the o.d. system

$$(20) \quad -Z' = \mathcal{L}_K^* Z \quad \text{on } [0, T], \quad Z(0) = Z_0 \in \mathbb{R}^{2K}.$$

From the **observability inequality** for system (19) we deduce the **unique continuation property** for the solutions to (20):

$$\boxed{B_K^* Z(\cdot) = 0 \quad \text{in } (0, T) \implies Z \equiv 0.}$$

5. Boundary controllability of a 2×2 linear system

In particular system

$$Y' = \mathcal{L}_K Y + B_K v \quad \text{on } [0, T], \quad Y(0) = Y_0 \in \mathbb{R}^{2K}.$$

is exactly controllable at time T . Then $\boxed{\text{rank} [\mathcal{L}_K \mid B_K] = 2K}$.

We deduce that \mathcal{L}_K^* cannot have eigenvalues with **geometric multiplicity** 2 or greater.

But $\theta = -d_1 \lambda_k = -d_2 \lambda_j$ is an eigenvalue of \mathcal{L}_K^* with two linearly independent eigenvectors $V_1, V_2 \in \mathbb{R}^{2K}$ given by:

$$\begin{cases} V_1 = (V_{1,\ell})_{1 \leq \ell \leq K}, & V_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } V_{1,\ell} = 0 \quad \forall \ell \neq k, \\ V_2 = (V_{2,\ell})_{1 \leq \ell \leq K}, & V_{2,j} = \begin{pmatrix} \frac{1}{\lambda_j(d_1 - d_2)} \\ 0 \end{pmatrix} \text{ and } V_{2,\ell} = 0 \quad \forall \ell \neq j. \blacksquare \end{cases}$$

The result has been proved in [[FERNÁNDEZ-CARA, G.-B., DE TERESA](#)], J. Funct. Anal. (2010).

5. Boundary controllability of a 2×2 linear system

Conclusion: **First difference with scalar problems**

distributed controllability \neq **boundary controllability**.

Even if System (14) is very close to System (18), their controllability properties are **strongly different**:

- System (14) (**distributed control**): We have obtained a complete characterization of the null controllability property **in the constant case** (and even, a distributed Carleman estimate for the **adjoint problem** (17)).
- System (18) (**boundary control**): The system is not null controllable if $d_1 \lambda_k = d_2 \lambda_j$ for some $k, j \geq 1$.

*The same **non-scalar parabolic problem** can be controlled to zero with distributed controls supported on an interval ω and, however, the **null controllability result** fails when the control acts on a part of the boundary.*

5. Boundary controllability of a 2×2 linear system

$$(18) \quad \begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad d_1 \neq d_2, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Remark

- Again, System (18) is always null controllable at time T if we exert **two independent controls** at the same point. In this case, equivalence between distributed and boundary controllability (as in the **scalar case**; see Theorem 1).
- If $d_1 \neq d_2$, one has: “System (18) is approximately controllable at time T $\iff \sqrt{d_1/d_2} \notin \mathbb{Q}$ ”. ■

5. Boundary controllability of a 2×2 linear system

$$(19) \quad \begin{cases} -\varphi_t = D\varphi_{xx} + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \{0, \pi\} \times (0, T), \quad \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi). \end{cases}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad d_1 \neq d_2, \quad \text{and} \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Boundary approximate controllability

“System (18) is approximately controllable at time $T \iff \sqrt{d_1/d_2} \notin \mathbb{Q}$ ”.

What does this condition mean???: The eigenvalues of the operator

$\mathcal{R}^*\Phi = D\Phi_{xx} + A^*\Phi$ are

$$\{-d_1k^2\}_{k \geq 1} \cup \{-d_2i^2\}_{i \geq 1}.$$

Then, $\sqrt{d_1/d_2} \notin \mathbb{Q} \iff$ the eigenvalues of \mathcal{R}^* are **simple**.

5. Boundary controllability of a 2×2 linear system

$$(18) \quad \begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases} \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Second difference with scalar problems

Null controllability: Assume $\sqrt{d_1/d_2} \notin \mathbb{Q}$. Is System (18) null controllable at time T ? i.e., are **approximate** controllability and **null** controllability equivalent for System (18)?

We will see that the answer is **negative**.

approximate controllability $\not\equiv$ **null controllability**.

(See also [[AMMAR-KHODJA, BENABDALLAH, DUPAIX, KOSTINE](#)], ESAIM:COCV (2005) for some abstract non-scalar parabolic systems).

6. A generalization: Cascade systems

6. A generalization: Cascade systems

We consider the linear parabolic system

$$\left\{ \begin{array}{l} \partial_t y_1 + L_0^1(t)y_1 + \sum_{j=1}^n C_{1j} \cdot \nabla y_j + \sum_{j=1}^n a_{1j}y_j = v1_\omega \quad \text{in } Q_T = \Omega \times (0, T), \\ \partial_t y_2 + L_0^2(t)y_2 + \sum_{j=1}^n C_{2j} \cdot \nabla y_j + \sum_{j=1}^n a_{2j}y_j = 0 \quad \text{in } Q_T, \\ \dots \\ \partial_t y_n + L_0^n(t)y_n + \sum_{j=1}^n C_{nj} \cdot \nabla y_j + \sum_{j=1}^n a_{nj}y_j = 0 \quad \text{in } Q_T, \\ y_i = 0 \text{ on } \Sigma_T = \partial\Omega \times (0, T), \quad y_i(\cdot, 0) = y_0^i \text{ in } \Omega, \quad 1 \leq i \leq n, \end{array} \right.$$

where $a_{ij} = a_{ij}(x, t) \in L^\infty(Q_T)$, $C_{ij} = C_{ij}(x, t) \in L^\infty(Q_T; \mathbb{R}^N)$ ($1 \leq i, j \leq n$), $y_0^i \in L^2(\Omega)$ ($1 \leq i \leq n$) and $L_0^k(t)$ is, for every $1 \leq k \leq n$, the second order

operator $L_0^k(t)y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}^k(x, t) \frac{\partial y}{\partial x_j} \right)$ where α_{ij}^k satisfy (3) and (4) for

every k .

6. A generalization: Cascade systems

Objective

Controllability properties of the system: n equations controlled with a **unique** distributed control.

Equivalently, the previous system can be written as

$$(21) \quad \begin{cases} \partial_t y + \widehat{L}(t)y + C \cdot \nabla y + Ay = Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\widehat{L}(t)$ is the **matrix operator** given by $\widehat{L}(t) = \text{diag}(L_0^1(t), \dots, L_0^n(t))$, $y = (y_i)_{1 \leq i \leq n}$ is the state and $\nabla y = (\nabla y_i)_{1 \leq i \leq n}$, and where

$$\begin{cases} y_0 = (y_0^i)_{1 \leq i \leq n} \in L^2(\Omega; \mathbb{R}^n), & A(\cdot, \cdot) = (a_{ij}(\cdot, \cdot))_{1 \leq i, j \leq n} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n)), \\ C(\cdot, \cdot) = (C_{ij}(\cdot, \cdot))_{1 \leq i, j \leq n} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^{N_n})) \text{ and } B \equiv e_1 = (1, 0, \dots, 0)^* \end{cases}$$

are given. Let us observe that, for each $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and $v \in L^2(Q_T)$, System (21) admits a **unique weak solution**

$$y \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^n)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^n)).$$

6. A generalization: Cascade systems

By **cascade system** we mean that matrices A and C have the following structure:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n-1} & a_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ 0 & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{nn} \end{pmatrix}$$

with $a_{ij} \in L^\infty(Q_T)$ and $C_{ij} \in L^\infty(Q_T; \mathbb{R}^N)$ and the coefficients $a_{i,i-1}$ satisfy

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0, T), \quad \forall i : 2 \leq i \leq n,$$

with $\omega_0 \subseteq \omega$ a new open subset.

Remark

It is natural to assume that $a_{i,i-1} \neq 0$ for any $i : 2 \leq i \leq n$. The previous assumption is **stronger** but will provide the controllability result. ■

6. A generalization: Cascade systems

In this case, the corresponding **adjoint problem** has the form

$$\left\{ \begin{array}{l} -\partial_t \varphi_i + L_0^i(t) \varphi_i - \sum_{j=1}^i [\nabla \cdot (C_{ji} \varphi_j) - a_{ji} \varphi_j] = -a_{i+1,i} \varphi_{i+1} \quad \text{in } Q_T, \\ \dots \quad \quad \quad (1 \leq i \leq n-1), \\ -\partial_t \varphi_n + L_0^n(t) \varphi_n - \sum_{j=1}^n [\nabla \cdot (C_{jn} \varphi_j) - a_{jn} \varphi_j] = 0 \quad \text{in } Q_T, \\ \varphi_i = 0 \text{ on } \Sigma_T, \quad \varphi_i(\cdot, T) = \varphi_{i,T} \text{ in } \Omega, \quad 1 \leq i \leq n, \end{array} \right.$$

where $\varphi_{i,T} \in L^2(\Omega)$ ($1 \leq i \leq n$). Again, the **null controllability** of System (21) (with L^2 -controls) at time T is equivalent to the existence of a constant $C_T > 0$ such that the so-called **observability inequality**

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \leq C_T \iint_{\omega \times (0, T)} |\varphi_1(x, t)|^2$$

holds for every solution $\varphi = (\varphi_1, \dots, \varphi_n)^*$ to the **adjoint problem**.

6. A generalization: Cascade systems

Theorem

Under the previous assumptions, let $M_0 = \max_{2 \leq i \leq n} \|a_{i,i-1}\|_\infty$. Then, there exist a positive function $\alpha_0 \in C^2(\bar{\Omega})$ (only depending on Ω and ω_0), two positive constants C_0 and σ_0 (only depending on Ω , ω_0 , c_0 , M_0 and d) and $l \geq 0$ (only depending on n) such that, for every $\varphi_T \in L^2(Q_T; \mathbb{R}^n)$, the solution φ to the **adjoint problem** satisfies

$$\sum_{i=1}^n \mathcal{I}(d + 3(n - i), \varphi_i) \leq C_0 s^{d+l} \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \gamma(t)^{d+l} |\varphi_1|^2,$$

$\forall s \geq s_0 = \sigma_0 \left[T + T^2 + T^2 \max_{i \leq j} \left(\|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}} \right) \right]$. In the

previous inequality, $\gamma(t) = t^{-1}(T-t)^{-1}$, $\alpha(x, t) = \alpha_0(x)/t(T-t)$ and $\mathcal{I}(d, z)$ is given in Lemma 2.3 (with α instead of β).

6. A generalization: Cascade systems

Combining the previous result and **energy inequalities** satisfied by the solutions of the **adjoint system** it is possible to prove an **observability inequality** for the **adjoint system** (as in the scalar case). Summarizing, we get

Corollary

Under assumptions of the previous result, there exists a positive constant C (only depending on Ω , ω , n , c_0 and M_0) such that for every $y_0 \in L^2(\Omega; \mathbb{R}^n)$ there is a control $v \in L^2(\Omega)$ which satisfies

$$\|v\|_{L^2(Q_T)}^2 \leq e^{C\mathcal{H}} \|y_0\|_{L^2(\Omega; \mathbb{R}^n)}^2,$$

and $y(\cdot, T) = 0$ in Ω , with y the solution to (21) associated to y_0 and v . In the previous inequality, \mathcal{H} is given by

$$\mathcal{H} \equiv 1 + T + \frac{1}{T} + \max_{i \leq j} \left(\|a_{ij}\|_{\infty}^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_{\infty}^{\frac{2}{3(j-i)+1}} + T (\|a_{ij}\|_{\infty} + \|C_{ij}\|_{\infty}^2) \right).$$

6. A generalization: Cascade systems

Sketch of the proof of Theorem 6.1: Given $\omega_0 \subset \omega$, we choose $\omega_1 \subset\subset \omega_0$. Let $\alpha_0 \in C^2(\overline{\Omega})$ be the function provided by Lemma 2.3 and associated to Ω and $\mathcal{B} \equiv \omega_1$. We will do the proof in two steps:

Step 1. Let φ be the solution to **adjoint system** associated to φ_T . Each component satisfies

$$-\partial_t \varphi_i + L_0^i(t) \varphi_i = \sum_{j=1}^i [\nabla \cdot (C_{ji} \varphi_j) - a_{ji} \varphi_j] - a_{i+1,i} \varphi_{i+1}.$$

We begin applying inequality (11) with $\mathcal{B} = \omega_1$ to each function φ_i with $L_0 \equiv L_0^i$, $d = d + 3(n - i)$ and the corresponding right-hand side. Now if we take

$$s \geq s_0 = \sigma_0 \left(T + T^2 + T^2 \max_{i \leq j} \left(\|a_{ij}\|_{\infty}^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_{\infty}^{\frac{2}{3(j-i)+1}} \right) \right),$$

with $\sigma_0 = \sigma_0(\Omega, \omega_0, c_0, M_0) > 0$, we obtain the existence of a positive constants $C_1 = C_1(\Omega, \omega_0, c_0, M_0)$ such that if $s \geq s_0$, then

6. A generalization: Cascade systems

$$\sum_{i=1}^n \mathcal{I}(d + 3(n - i), \varphi_i) \leq C_1 \sum_{i=1}^n s^{s+3(n-i)} \iint_{\omega_1 \times (0, T)} e^{-2s\alpha\gamma(t)^{s+3(n-i)}} |\varphi_i|^2.$$

Step 2. Thanks to the assumption

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0, T), \quad \forall i : 2 \leq i \leq n,$$

with $\omega_0 \subseteq \omega$ an open subset, and the cascade structure

$$a_{i,i-1}\varphi_i = \partial_t \varphi_{i-1} - L_0^{i-1}(t)\varphi_{i-1} + \sum_{j=1}^{i-1} [\nabla \cdot (C_{j,i-1}\varphi_j) - a_{j,i-1}\varphi_{i-1}] \text{ in } Q_T,$$

can eliminate the local terms for $2 \leq i \leq n$. In order to carry this process out, we will need the following result:

6. A generalization: Cascade systems

Lemma

Under assumptions of Theorem 6.1 and given $l \in \mathbb{N}$, $\varepsilon > 0$, $k \in \{2, \dots, n\}$ and two open sets \mathcal{O}_0 and \mathcal{O}_1 such that $\omega_1 \subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset \omega_0$, there exist a constant C_k (only depending on Ω , \mathcal{O}_0 , \mathcal{O}_1 , c_0 and M_0) and $l_{kj} \in \mathbb{N}$, $1 \leq j \leq k-1$ (only depending on l , n , k and j), such that, if $s \geq s_0$, one has

$$s^l \iint_{\mathcal{O}_1 \times (0, T)} e^{-2s\alpha} \gamma(t)^l |\varphi_k|^2 \leq \varepsilon [\mathcal{I}(d + 3(n - k), \varphi_k) + \mathcal{I}(d + 3(n - k - 1), \varphi_{k+1})] \\ + C_k \left(1 + \frac{1}{\varepsilon}\right) \sum_{j=1}^{k-1} s^{l_{kj}} \iint_{\mathcal{O}_0 \times (0, T)} e^{-2s\alpha} \gamma(t)^{l_{kj}} |\varphi_j|^2.$$

(In this inequality we have taken $\varphi_{k+1} \equiv 0$ when $k = n$).

The proof of Theorem 6.1 is a consequence of this Lemma 6.3. For the details, see [DE TERESA], Comm. PDE (2000), [G.-B., PÉREZ-GARCÍA], Asymp. Anal. (2006) and [G.-B., DE TERESA], Port. Math. (2010).

6. A generalization: Cascade systems

Remark

- 1 **Cascade systems** appear in the context of existence of **insensitizing controls** for a scalar parabolic equation: Equivalent to a null controllability result for a 2×2 parabolic system ($n = 2$) with one equation forward in time and the other one backward. The coupling coefficient a_{21} is $1_{\mathcal{O}}$ with $\mathcal{O} \subseteq \Omega$ an open set and $\mathcal{O} \cap \omega \neq \emptyset$.
- 2 The previous proof uses the assumption

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0, T), \forall i : 2 \leq i \leq n,$$

in a crucial way. When $a_{i,i-1}$ are constant, this assumption is **necessary**. Is this condition **necessary** in the general case??? **No**.

- 3 Is it possible to provide a **necessary** and **sufficient** (**Kalman condition**) condition for the null controllability of non-scalar systems? **YES** in some **constant coefficient systems**.

6. A generalization: Cascade systems

Some additional references

- 1 [L. MANIAR ET AL.](#), Controllability results for degenerate parabolic cascade systems.
- 2 [M. DUPREZ, P. LISSY](#), Controllability results for parabolic systems with first order coupling terms.

7. The Kalman condition for a class of parabolic systems. Distributed controls

7. The Kalman condition for a class of parabolic systems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$, with boundary $\partial\Omega$ of class C^2 . Let $\omega \subseteq \Omega$ be an open subset and let us fix $T > 0$.

For $n, m \in \mathbb{N}$ we consider the following autonomous $n \times n$ parabolic system

$$(22) \quad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ and $D = \text{diag}(d_1, d_2, \dots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$. We assume that L_0 is the self-adjoint second order elliptic operator:

$$L_0 y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x) \frac{\partial y}{\partial x_j} \right)$$

with coefficients satisfying (3) and (4). Finally, $y_0 \in L^2(\Omega; \mathbb{R}^n)$ is given and $v \in L^2(Q_T; \mathbb{R}^m)$ is the control (m **distributed controls**).

4. The Kalman condition for a class of parabolic systems

$$(22) \quad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega. \end{cases}$$

Remark

This problem is **well posed**: For any $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and $v \in L^2(Q_T; \mathbb{R}^m)$, problem (22) has a **unique solution**

$$y \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^n)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^n)).$$

Remark

We want to control the whole system (n **equations**) with m **controls**. The most interesting case is $m < n$ or even $m = 1$.

Difficulties:

- 1 In general $m < n$.
- 2 D is not the identity matrix. ■

4. The Kalman condition for a class of parabolic systems

The adjoint problem:

$$(23) \quad \begin{cases} -\partial_t \varphi = (-DL_0 + A^*)\varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega, \end{cases}$$

where $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$. Then, the **exact controllability to the trajectories** of system (22) **is equivalent** to the existence of $C_T > 0$ such that, for every $\varphi_0 \in L^2(\Omega; \mathbb{R}^n)$, the solution $\varphi \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$ to the adjoint system (23) satisfies the **observability inequality**:

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C_T \iint_{\omega \times (0, T)} |B^* \varphi(x, t)|^2.$$

7. The Kalman condition for a class of parabolic systems

We come back to System (22):

$$(22) \quad \begin{cases} \partial_t y + DL_0 y = Ay + Bv \mathbf{1}_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ and $D = \text{diag}(d_1, d_2, \dots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$. Now we assume that L_0 is the self-adjoint second order elliptic operator:

$$L_0 y = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(\alpha_{ij}(x) \frac{\partial y}{\partial x_j} \right)$$

with coefficients satisfying (3) and (4). Finally, $y_0 \in L^2(\Omega; \mathbb{R}^n)$ is given and $v \in L^2(Q_T; \mathbb{R}^m)$ is the control (m **distributed controls**).

7. The Kalman condition for a class of parabolic systems

Let us consider $\{\lambda_k\}_{k \geq 1}$ the sequence of eigenvalues for L_0 with homogeneous Dirichlet boundary conditions and $\{\phi_k\}_{k \geq 0}$ the corresponding normalized eigenfunctions.

Theorem (A Necessary Condition)

If system (22) is **null controllable** at time T **then**

$$(24) \quad \text{rank} [-\lambda_k D + A \mid B] = n, \quad \forall k \geq 1.$$

where

$$[-\lambda_k D + A \mid B] = [B, (-\lambda_k D + A)B, (-\lambda_k D + A)^2 B, \dots, (-\lambda_k D + A)^{n-1} B].$$

Proof: Reasoning by contradiction: $\exists k \geq 1$ such that

$\text{rank} [-\lambda_k D + A \mid B] < n$. Then the o.d.s. $-Z' = (-\lambda_k D + A^*)Z$ in $(0, T)$, is not B^* -observable at time T .

7. The Kalman condition for a class of parabolic systems

There exists $Z_0 \in \mathbb{R}^n$, $Z_0 \neq 0$, such that the solution Z to the previous system satisfies $B^*Z(\cdot) = 0$ on $(0, T)$. But $\varphi(x, t) = Z(t)\phi_k(x)$ is the solution to **adjoint problem**

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega, \end{cases}$$

associated to $\varphi_0(x) = Z_0 \phi_k \neq 0$ and $B^* \varphi(\cdot, \cdot) \equiv 0$ in Q_T . Then, the **observability inequality**

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C_T \iint_{\omega \times (0, T)} |B^* \varphi(x, t)|^2,$$

fails and the system is not **null controllable** at time T . ■

Remark

If condition (24) is not satisfied, then system (22) is neither **approximately controllable** nor **null controllable** at time T (for any $T > 0$) even if $\omega \equiv \Omega$. ■

7. The Kalman condition for a class of parabolic systems

Question:

Is condition (24) $\text{rank} [-\lambda_k D + A \mid B] = n, \forall k \geq 1$, a **sufficient condition** for the **null controllability** of system (22)???

Let us now introduce the **unbounded matrix operator**

7. The Kalman condition for a class of parabolic systems

Question:

Is condition (24) $\text{rank} [-\lambda_k D + A \mid B] = n, \forall k \geq 1$, a **sufficient condition** for the **null controllability** of system (22)???

Let us now introduce the **unbounded matrix operator**

$$\mathcal{K} = [DL_0 + A \mid B] = [B, (-DL_0 + A)B, \dots, (-DL_0 + A)^{n-1}B],$$

$$\begin{cases} \mathcal{K} : D(\mathcal{K}) \subset L^2(\Omega; \mathbb{R}^{nm}) \rightarrow L^2(\Omega; \mathbb{R}^n), \text{ with} \\ D(\mathcal{K}) := \{y \in L^2(\Omega; \mathbb{R}^{nm}) : \mathcal{K}y \in L^2(\Omega; \mathbb{R}^n)\}. \end{cases}$$

Then,

Proposition

$\ker \mathcal{K}^* = \{0\}$ **if and only if** condition (24), $\text{rank} [-\lambda_k D + A \mid B] = n, \forall k \geq 1$, holds.

7. The Kalman condition for a class of parabolic systems

$$(22) \quad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

Theorem (**Kalman condition**)

System (22) is **exactly controllable to trajectories** at time T **if and only if**

System (22) is **approximately controllable** at time T **if and only if**

$$\ker \mathcal{K}^* = \{0\} \quad (\iff \text{rank} [-\lambda_k D + A \mid B] = n, \forall k \geq 1).$$

Remark

One can prove, either there exists $k_0 \geq 1$ such that

$$\text{rank} [-\lambda_k D + A \mid B] = n, \quad \forall k \geq k_0$$

or

$$\text{rank} [-\lambda_k D + A \mid B] < n, \quad \forall k \geq 1.$$

7. The Kalman condition for a class of parabolic systems

Controllability (outside a finite dimensional space) **if and only if** the algebraic Kalman condition $\text{rank} [-\lambda_k D + A \mid B] = n$ is satisfied for one frequency $k \geq 1$.

Remark

System (22) can be **exactly controlled to the trajectories** with one control force ($m = 1$ and $B \in \mathbb{R}^n$) even if $A \equiv 0$. Indeed, let us assume that $B = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n$. Then,

$$[(-\lambda_k D + A) \mid B] = \begin{bmatrix} b_1 & (-\lambda_k d_1) b_1 & \cdots & (-\lambda_k d_1)^{n-1} b_1 \\ b_2 & (-\lambda_k d_2) b_2 & \cdots & (-\lambda_k d_2)^{n-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & (-\lambda_k d_n) b_n & \cdots & (-\lambda_k d_n)^{n-1} b_n \end{bmatrix} \in \mathcal{L}(\mathbb{R}^n),$$

and (24) holds **if and only if** $b_i \neq 0$ for every i and d_i are **distinct**. ■

7. The Kalman condition for a class of parabolic systems

Idea of the proof: We have proved the **necessary condition**. Therefore, let us prove that $\boxed{\text{rank} [-\lambda_k D + A \mid B] = n}$, for any k , is a **sufficient condition** for the null controllability at time T of the system.

Then, the objective is to prove the **observability inequality**:

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0, T)} |B^* \varphi(x, t)|^2,$$

for the solutions to the **adjoint problem**.

To this end we use two arguments:

- Prove a global Carleman estimate for a **scalar parabolic equation of order n in time**.
- Prove a **coercivity** property for the Kalman operator \mathcal{K} .

7. The Kalman condition for a class of parabolic systems

Let us fix $\varphi_0 \in D(L_0^i), \forall i \geq 0$ and consider φ the corresponding solution to the **adjoint system** (23)

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega. \end{cases}$$

Let us take $\Phi = \sum_{i=1}^n a_i \varphi_i$, with $a_i \in \mathbb{R} (1 \leq i \leq n)$. Then, Φ is a regular solution ($L_0^i \partial_t^j \Phi \in L^2(Q_T), \forall i, j$) to the **linear parabolic scalar equation of order n** in time

$$\begin{cases} \det(I_d \partial_t - DL_0 + A^*) \Phi = 0 & \text{in } Q_T, \\ L_0^i \Phi = 0 & \text{on } \Sigma_T, \quad \forall i \geq 0. \end{cases}$$

The key point is to prove a Carleman inequality for the solutions to the previous problem. Fix $\omega_0 \subset\subset \omega$ a nonempty open subset. Recall Lemmas 2.3 and 2.4:

7. The Kalman condition for a class of parabolic systems

Lemma

There exist a $\alpha_0 \in C^2(\bar{\Omega})$ (positive), and two constants $C_0, \sigma_0 > 0$ (only depending on Ω, ω_0 and d) s.t.

$$\left\{ \begin{array}{l} \mathcal{I}_1(d, \phi) \equiv \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-4} (|\phi_t|^2 + |L_0\phi|^2) \\ + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-2} |\nabla\phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |\phi|^2 \\ \leq C_0 \left(\iint_{\omega_0 \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^d |\phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^{d-3} |\phi_t \pm L_0\phi|^2 \right), \end{array} \right.$$

$\forall s \geq s_0 = \sigma_0(\Omega, \omega)(T + T^2), \forall \phi \in L^2(0, T; H_0^1(\Omega))$ s.t. $\phi_t \pm L_0\phi \in L^2(Q_T)$.

$$\boxed{\gamma(t) = t^{-1}(T-t)^{-1}}, \quad \boxed{\alpha(x, t) = \alpha_0(x)/t(T-t)}.$$

7. The Kalman condition for a class of parabolic systems

Theorem

Let $n, k_1, k_2 \in \mathbb{N}$ and $d \in \mathbb{R}$. There exist two constants C and σ (only depending on $\Omega, \omega, n, D, A, k_1, k_2$ and d), and $r_0 = r_0(n) \in \mathbb{N}$ such that

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \mathcal{J}(d - 4(i+j), L_0^i \partial_t^j \Phi) \leq C \iint_{\omega \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{3+r_0} |\Phi|^2, \quad ,$$

$\forall s \geq s = \sigma(\Omega, \omega)(T + T^2)$, Φ solution to the previous problem and

$$\begin{aligned} \mathcal{J}(\tau, z) &:= \mathcal{I}_1(\tau + 3(n-1), z) + \sum_{i=1}^n \mathcal{I}_1(\tau + 3(n-2), P_i z) \\ &\quad + \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \dots < i_p \leq n} \mathcal{I}_1(\tau + 3(n-p-1), P_{i_p} \cdots P_{i_1} z). \\ (P_i &\equiv \partial_t - d_i L_0) \end{aligned}$$

7. The Kalman condition for a class of parabolic systems

Sketch of the proof: We will give the main ideas in the case $k_1 = k_2 = 0$. If we use the notation $P_i \equiv \partial_t - d_i L_0$ ($1 \leq i \leq n$), one has:

$$\begin{aligned} \det(I_d \partial_t - DL_0 + A^*) &\equiv P_n \cdots P_1 + \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \cdots < i_p \leq n} b_{i_1, \dots, i_p} P_{i_1} \cdots P_{i_p} \\ &\quad + \sum_{i=1}^n b_i P_i + b := P_n \cdots P_1 - F, \end{aligned}$$

with $b_{i_1, \dots, i_p}, b_i, b \in \mathbb{R}$ only depending on D and A .

We have a function Φ s.t. $L_0^i \partial_t^j \Phi \in L^2(Q_T)$, $\forall i, j$, and it is solution to

$$\begin{cases} \det(I_d \partial_t - DL_0 + A^*) \Phi = 0 & \text{in } Q_T, \\ L_0^i \Phi = 0 & \text{on } \Sigma, \quad \forall i \geq 0. \end{cases}$$

In particular, $P_n \cdots P_1 \Phi = F(\Phi)$ in Q_T .

7. The Kalman condition for a class of parabolic systems

In particular, $P_n \cdots P_1 \Phi = F(\Phi)$ in Q_T . We rewrite the order- n equation as a system performing the change of variables:

$$\begin{cases} \psi_1 := \Phi, \\ \psi_i := P_{i-1} \psi_{i-1} \equiv (\partial_t - d_{i-1}) \psi_{i-1}, \quad 2 \leq i \leq n. \end{cases}$$

Then, $\Psi = (\psi_1, \psi_2, \dots, \psi_n)^*$ satisfies the **cascade system**

$$\begin{cases} (\partial_t - d_1 L_0) \psi_1 = \psi_2 & \text{in } Q_T, \\ (\partial_t - d_2 L_0) \psi_2 = \psi_3 & \text{in } Q_T, \\ \vdots \\ (\partial_t - d_n L_0) \psi_n = F(\Phi) & \text{in } Q_T, \\ \psi_i = 0 \text{ on } \Sigma_T, & \forall i : 1 \leq i \leq n. \end{cases}$$

We can apply Theorem 6.1 (cascade systems) and obtain:

7. The Kalman condition for a class of parabolic systems

We can apply Theorem 6.1 and obtain (cascade systems) ($d \in \mathbb{R}$ is given):

$$\sum_{i=1}^n \mathcal{I}_1(d + 3(n - i), \psi_i) \leq C_0 \left(\iint_{\omega \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\psi_1|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |F(\Phi)|^2 \right),$$

$\forall s \geq s_0 = \sigma_0 (T + T^2)$ with $r_0 = r_0(n)$ and

$$\mathcal{I}_1(d, z) \equiv \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d \{ [s\gamma(t)]^{-4} (|\partial_t z|^2 + |L_0 z|^2) + [s\gamma(t)]^{-2} |\nabla z|^2 + |z|^2 \}.$$

Coming to the original variables, one has

$$\begin{aligned} & \mathcal{I}_1(d + 3(n - 1), \Phi) + \sum_{i=2}^n \mathcal{I}_1(d + 3(n - i), P_{i-1} \cdots P_1 \Phi) \\ & \leq C_0 \left(\iint_{\omega \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\Phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |F(\Phi)|^2 \right). \end{aligned}$$

7. The Kalman condition for a class of parabolic systems

We can reproduce the previous argument for a general permutation Π of the set $\{1, 2, \dots, n\}$, taking

$$\begin{cases} \psi_1 := \Phi, \\ \psi_i := P_{\Pi(i-1)}\psi_{i-1} \equiv (\partial_t - d_{\Pi(i-1)})\psi_{\Pi(i-1)}, \quad 2 \leq i \leq n. \end{cases}$$

Thus,

$$\begin{aligned} & \mathcal{I}_1(d + 3(n - 1), \Phi) + \sum_{i=2}^n \mathcal{I}_1(d + 3(n - i), P_{\Pi(i-1)} \cdots P_{\Pi(1)}\Phi) \\ & \leq C_0 \left(\iint_{\omega \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\Phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |F(\Phi)|^2 \right), \end{aligned}$$

$\forall s \geq s_0 = \sigma_0 (T + T^2)$. Adding all these inequalities (for any permutation Π) with $d = 3$, we get

7. The Kalman condition for a class of parabolic systems

Adding all these inequalities (for any permutation Π) with $d = 3$, we get

$$\mathcal{J}(d, \Phi) \leq C \left(\iint_{\omega \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\Phi|^2 + \iint_{Q_T} e^{-2s\alpha} [s\gamma(t)]^d |F(\Phi)|^2 \right),$$

$\forall s \geq s_0 = \sigma_0 (T + T^2)$ ($\mathcal{J}(\tau, z)$ given in the statement of Theorem 10 and

$$F(\Phi) = \sum_{p=2}^{n-1} \sum_{1 \leq i_1 < \dots < i_p \leq n} b_{i_1, \dots, i_p} P_{i_1} \dots P_{i_p} \Phi + \sum_{i=1}^n b_i P_i \Phi + b \Phi).$$

From these expressions, it is possible to **absorb** the last term of the previous inequality and obtain

$$\mathcal{J}(d, \Phi) \leq C \iint_{\omega \times (0, T)} e^{-2s\alpha} [s\gamma(t)]^{d+r_0} |\Phi|^2,$$

for a new constant C , with $s \geq s = \sigma (T + T^2)$. This ends the proof in the case $k_1 = k_2 = 0$. ■

7. The Kalman condition for a class of parabolic systems

Remark

Theorem 10 is, in fact, a Carleman inequality for the regular solutions Φ to the **linear parabolic scalar equation of order n** in time

$$\begin{cases} \det(I_d \partial_t - DL_0 + A^*) \Phi = 0 & \text{in } Q_T, \\ L_0^i \Phi = 0 & \text{on } \Sigma, \quad \forall i \geq 0. \end{cases}$$

7. The Kalman condition for a class of parabolic systems

Conclusion

If φ is a regular solution to the **adjoint problem**

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega, \end{cases}$$

then, any linear combination $\Phi = \sum_{i=1}^n a_i \varphi_i$ satisfies Theorem 10. In particular any component of $B^* \varphi$. ■

7. The Kalman condition for a class of parabolic systems

Conclusion

If φ is a regular solution to the **adjoint problem**

$$\begin{cases} -\partial_t \varphi + DL_0 \varphi = A^* \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } \Omega, \end{cases}$$

then, any linear combination $\Phi = \sum_{i=1}^n a_i \varphi_i$ satisfies Theorem 10. In particular any component of $B^* \varphi$. ■

Recall $\mathcal{K} = [DL_0 + A \mid B] = [B, (-DL_0 + A)B, \dots, (-DL_0 + A)^{n-1}B]$, then

$$\begin{aligned} \mathcal{K}^* \varphi(\cdot, t) &= [B^* \varphi, B^* (-DL_0 + A^*) \varphi, \dots, B^* (-DL_0 + A^*)^{n-1} \varphi]^{tr}(\cdot, t) \\ &= [B^* \varphi, -\partial_t (B^* \varphi), \dots, (-1)^{n-1} \partial_t^{n-1} (B^* \varphi)]^{tr}(\cdot, t) \in \mathbb{R}^{nm}. \end{aligned}$$

We apply Theorem 10 with $k_1 = n - 1$ and $k_2 = k \geq 0$. Then, after some computations, we deduce ($d = 3$)

7. The Kalman condition for a class of parabolic systems

Then, after some computations, we deduce ($d = 3$)

$$\int_0^T e^{\frac{-2sM_0}{t(T-t)}} [s\gamma(t)]^3 \|L_0^k \mathcal{K}^* \varphi\|_{L^2(\Omega)^{nm}}^2 \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} [s\gamma(t)]^{3+r_0} |B^* \varphi|^2$$

for every $s \geq \sigma(T + T^2)$. In this inequality, $M_0 = \max_{\bar{\Omega}} \alpha_0$ and $r_0 \geq 0$ is an integer only depending on n .

Remark

The previous inequality is a **partial observability estimate**. It is valid even if the Kalman condition does not hold, i.e., even if $\ker \mathcal{K}^* \neq \{0\}$. ■

7. The Kalman condition for a class of parabolic systems

The **coercivity** property of \mathcal{K}^* :

Theorem

Assume that $\ker \mathcal{K}^* = \{0\}$ and consider $k = (n-1)(2n-1)$. Then there exists $C > 0$ such that if $z \in L^2(\Omega)^n$ satisfies $\mathcal{K}^* z \in D(L_0^k)^{nm}$, one has

$$\|z\|_{L^2(\Omega)^n}^2 \leq C \|L_0^k \mathcal{K}^* z\|_{L^2(\Omega)^{nm}}^2.$$

So, from the previous inequality we get

$$\int_0^T e^{\frac{-2sM_0}{t(T-t)}} [s\gamma(t)]^3 \|\varphi\|_{L^2(\Omega)^{nm}}^2 \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} [s\gamma(t)]^{3+r_0} |B^* \varphi|^2$$

and the **observability inequality**:

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0,T)} |B^* \varphi(x, t)|^2.$$

7. The Kalman condition for a class of parabolic systems

Summarizing

- 1 We have established a **Kalman condition**

$$\ker \mathcal{K}^* = \{0\}$$

which characterizes the controllability properties of system (22).

- 2 The **Kalman condition** for system (22) $\ker \mathcal{K}^* = \{0\}$ generalizes the **algebraic Kalman condition** $\ker[A \mid B]^* = \{0\}$ for o.d.s.
- 3 This **Kalman condition** is also equivalent to the **approximate controllability** of system (22) at time T . Again, **approximate** and **null controllability** are equivalent concepts for system (22).

7. The Kalman condition for a class of parabolic systems

References

- ① F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, M. G.-B., *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems*, *Differ. Equ. Appl.* **1** (2009), no. 3, 139–151.

$D = I_d$, $A = A(t)$ and $B = B(t)$.

- ② F. AMMAR-KHODJA, A. BENABDALLAH, C. DUPAIX, M. G.-B., *Kalman rank condition for the localized distributed controllability of a class of linear parabolic systems*, *J. Evol. Equ.* **9** (2009), no. 2, 267–291.

D diagonal matrix, A and B constant matrices.

- ③ E. FERNÁNDEZ-CARA, M. G.-B, L. DE TERESA, *Controllability of linear and semilinear non-diagonalizable parabolic systems*, *ESAIM Control Optim. Calc. Var.* **21** (2015), no. 4, 1178–1204.

D non-diagonalizable matrix with Jordan blocks of dimension ≤ 4 ,

A and B constant matrices.

7. The Kalman condition for a class of parabolic systems

Open problems

- Null controllability properties of

$$(22) \quad \begin{cases} \partial_t y + DL_0 y = A(t)y + B(t)v1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

when $A(t)$ and $B(t)$ depend on t (for instance, $A \in C^\infty([0, T]; \mathcal{L}(\mathbb{R}^n))$ and $B \in C^\infty([0, T]; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$) and $D = \text{diag}(d_1, d_2, \dots, d_n) \in \mathcal{L}(\mathbb{R}^n)$ with $d_i > 0$.

- Null controllability properties of

$$(22) \quad \begin{cases} \partial_t y + DL_0 y = Ay + Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

when A and B are constant matrices and D is a general non-diagonalizable matrix (definite positive).

8. The Kalman condition for a class of parabolic systems. Boundary controls

[AMMAR-KHODJA, BENABDALLAH, G.-B., DE TERESA], J. Math. Pures Appl. (2011).

8. The Kalman condition for a class of parabolic systems. Boundary controls

Let us consider the **boundary controllability problem**:

$$(25) \quad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

where $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$ are two given matrices and $y_0 \in H^{-1}(0, \pi; \mathbb{C}^n)$ is the initial datum. In system (25), $v \in L^2(0, T; \mathbb{C}^m)$ is the control function (to be determined).

Simpler problem: One-dimensional case and $D = Id$.

This problem has been studied in the case $n = 2$:

- E. FERNÁNDEZ-CARA, M. G.-B., L. DE TERESA, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. 259 (2010), no. 7, 1720–1758.

8. The Kalman condition for a class of parabolic systems.

Boundary controls

We consider again $\{\lambda_k\}_{k \geq 1}$ the sequence of **eigenvalues** for $-\partial_{xx}$ in $(0, \pi)$ with homogenous Dirichlet boundary conditions and $\{\phi_k\}_{k \geq 0}$ the corresponding **normalized eigenfunctions**:

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi).$$

Theorem ($n = 2, m = 1$)

Let $A \in \mathcal{L}(\mathbb{C}^2)$ and $B \in \mathbb{C}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A^* . Then (25) is **exactly controllable to the trajectories** at any time $T > 0$ **if and only if** $\text{rank } [A \mid B] = 2$ and

$$\lambda_k - \lambda_j \neq \mu_1 - \mu_2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j. \quad \blacksquare$$

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Distributed controllability and boundary controllability

- ① We proved that system

$$\begin{cases} y_t = y_{xx} + Ay + Bv1_\omega & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

is null controllable at time $T > 0$ **if and only if** $\boxed{\text{rank } [A \mid B] = 2}$.

- ② System

$$\begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

is null controllable at time $T > 0$ **if and only if** $\boxed{\text{rank } [A \mid B] = 2}$ and

$$\boxed{\lambda_k - \lambda_j \neq \mu_1 - \mu_2}.$$

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Remark ($n = 2, m = 1$)

For the previous **boundary controllability problem**, one has

- 1 A complete characterization of the **exact controllability to trajectories** at time T : **Kalman condition**.
- 2 **Boundary controllability** and **distributed controllability** are not equivalent
- 3 **Approximate controllability** \iff **null controllability**.

What happens if $n > 2$??

As we saw before, we will work in the following finite-dimensional space:

$$X_k = \left\{ \varphi_0 = \sum_{\ell=1}^k a_{\ell} \phi_{\ell} : a_{\ell} \in \mathbb{C}^n \right\} \subset H_0^1(0, \pi; \mathbb{C}^n).$$

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Adjoint Problem:

$$(26) \quad \begin{cases} -\varphi_t = \varphi_{xx} + A^* \varphi & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi), \end{cases}$$

with $\varphi_0 \in H_0^1(0, \pi; \mathbb{C}^n)$. Then, system (25) is **exactly controllable to trajectories** at time $T \iff$ for a constant $C > 0$ one has (**observability inequality**)

$$\|\varphi(\cdot, 0)\|_{H_0^1(0, \pi; \mathbb{C}^n)}^2 \leq C \int_0^T |B^* \varphi_x(0, t)|^2 dt.$$

Taking initial data in X_k , we deduce that an appropriate o.d. system in \mathbb{C}^{nk} also satisfies an **observability inequality**. Let us analyze this finite-dimensional system.

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Notation

For $k \geq 1$, we introduce $L_k = -\lambda_k I_d + A \in \mathcal{L}(\mathbb{C}^n)$ and the matrices

$$B_k = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad \mathcal{L}_k = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_k \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{nk}),$$

and let us write the Kalman matrix associated with the pair (\mathcal{L}_k, B_k) :

$$\mathcal{K}_k = [\mathcal{L}_k \mid B_k] = [B_k, \mathcal{L}_k B_k, \mathcal{L}_k^2 B_k, \dots, \mathcal{L}_k^{nk-1} B_k] \in \mathcal{L}(\mathbb{C}^{mnk}, \mathbb{C}^{nk}).$$

With this notation, the o.d. system associated to the **adjoint system** (26) for $\varphi_0 \in X_k$ is $\boxed{-Z' = \mathcal{L}_k^* Z \text{ on } (0, T), Z(T) = Z_0 \in \mathbb{C}^{nk}}$, and the solutions must be B_k^* -observable, i.e., $\text{rank } \mathcal{K}_k = nk$: **necessary condition**. One has:

8. The Kalman condition for a class of parabolic systems. Boundary controls

Theorem

Let us fix $A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$. Then, system (25) is **exactly controllable to trajectories** at time T if and only if

$$(27) \quad \text{rank } \mathcal{K}_k = nk, \quad \forall k \geq 1.$$

Remark

- 1 This result gives a complete characterization of the **exact controllability to trajectories** at time T : **Kalman condition**.
- 2 If for $k \geq 1$ one has $\text{rank } \mathcal{K}_k = nk$, then $\text{rank } [A \mid B] = n$ and system

$$\begin{cases} \partial_t y - \Delta y = Ay + Bv1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, \end{cases}$$

is **exactly controllable to trajectories** at time T . But $\text{rank } [A \mid B] = n$ does not imply condition (27). So **boundary controllability** and **distributed controllability** are not equivalent.

8. The Kalman condition for a class of parabolic systems. Boundary controls

Remark

Condition (27) is also a **necessary** and **sufficient condition** for the **boundary approximate controllability** of system (25). Then

Approximate controllability \iff **null controllability**.

Remark (n controls)

If $\boxed{\text{rank } B = n}$ (and thus $m \geq n$), then the pair (A, B) fulfills condition (27) and the system is **exactly controllable to trajectories** at time T . ■

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Remark (One control, $m = 1$)

When $m = 1$, the **Kalman condition** (27) is equivalent to $\text{rank } [A \mid B] = n$ and $\lambda_k - \lambda_l \neq \mu_i - \mu_j$ for any $k, l \in \mathbb{N}$ and $1 \leq i, j \leq p$ with $(k, i) \neq (l, j)$, where $\{\mu_i\}_{1 \leq i \leq p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^* . We generalize the results of [FERNÁNDEZ-CARA, G.-B., DE TERESA], J. Funct. Anal. (2010). ■

One control, $m = 1$

We have imposed two conditions:

- 1 $\text{rank } [A \mid B] = n$: System (25) is not **decoupled**.
- 2 $\lambda_k - \lambda_l \neq \mu_i - \mu_j$: The adjoint system can be written ($\mathcal{R}_0 = I_d \partial_{xx} + A^*$)

$$(26) \quad \begin{cases} -\varphi_t = \mathcal{R}_0 \varphi & \text{in } Q_T, \\ \varphi = 0 \text{ on } \Sigma_T, \quad \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi), \end{cases}$$

and the eigenvalues of \mathcal{R}_0 are **simple**. ■

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Before proving the result, let us analyze the **Kalman condition** (27)

$\text{rank } \mathcal{K}_k = nk, \forall k \geq 1:$

Proposition

Let us denote by $\{\mu_i\}_{1 \leq i \leq p} \subset \mathbb{C}$ the set of distinct eigenvalues of A^* . Then,

- ① There exists an integer $k_0 = k_0(A) \in \mathbb{N}$, only depending on A , such that,

$$\lambda_k - \lambda_l \neq \mu_i - \mu_j, \quad \forall k > k_0, l \geq 1, k \neq l, \text{ and } 1 \leq i, j \leq p.$$

- ② The following conditions are equivalent:

- (a) $\text{rank } \mathcal{K}_k = nk$ for every $k \geq 1$.
- (b) $\text{rank } \mathcal{K}_k = nk$ for every $k : 1 \leq k \leq k_0$.
- (c) $\text{rank } \mathcal{K}_{k_0} = nk_0$.

8. The Kalman condition for a class of parabolic systems. Boundary controls

Necessary implication. We reason as before: if $\text{rank } \mathcal{K}_k < nk$, for some $k \geq 1$, then the o.d.s.

$$-Z' = \mathcal{L}_k^* Z \quad \text{on } (0, T), \quad Z(T) = Z_0 \in \mathbb{C}^{nk}$$

is not B_k^* -observable on $(0, T)$, i.e., there exists $Z_0 \neq 0$ s.t. $B_k^* Z(t) = 0$ for every $t \in (0, T)$. From Z_0 it is possible to construct $\varphi_0 \in H_0^1(0, \pi; \mathbb{C}^n)$ with $\varphi_0 \neq 0$ such that the corresponding solution to the **adjoint problem** (27) satisfies

$$B^* \varphi_x(0, t) = 0 \quad \forall t \in (0, T).$$

As a consequence: The **unique continuation property** and the previous **observability inequality** for the **adjoint problem** fail:

Neither **approximate** nor **null controllability** at any T for system (25).

8. The Kalman condition for a class of parabolic systems. Boundary controls

Sufficient implication. For the proof we follow the ideas from

- H.O. FATTORINI, D.L. RUSSELL, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal. 43 (1971), 272–292.

Two “big” steps:

- (I) We reformulate the null controllability problem for system (25) as a **vector moment problem**.
- (II) Existence and bounds of a family **biorthogonal** to appropriate complex matrix exponentials.

8. The Kalman condition for a class of parabolic systems.

Boundary controls

(I) The **vector moment problem**: As in the scalar case, $\mathbf{v} \in L^2(0, T; \mathbb{C}^m)$ is a **null control** for system

$$(25) \quad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T, \\ y(0, \cdot) = B\mathbf{v}, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

(i.e., the solution y to (25) satisfies $y(\cdot, T) = 0$ in $(0, \pi)$) $\iff \mathbf{v}$ satisfies

$$-\langle y_0, \varphi(\cdot, 0) \rangle = \int_0^T (\mathbf{v}(t), B^* \varphi_x(0, t))_{\mathbb{C}^m} dt, \quad \forall \varphi_0 \in H_0^1(0, \pi; \mathbb{C}^n),$$

where φ is the solution to the **adjoint problem**

$$(26) \quad \begin{cases} -\varphi_t = \varphi_{xx} + A^* \varphi & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi). \end{cases}$$

8. The Kalman condition for a class of parabolic systems.

Boundary controls

(I) The **vector moment problem**:

Thus, the idea is to take firstly $\varphi_0 \in X_{k_0}$,

($X_{k_0} = \{\varphi_0 : \varphi_0 = \sum_{i=1}^{k_0} a_i \phi_i \text{ with } a_i \in \mathbb{C}^n\}$) and then $\varphi_0 = a\phi_k$, with $k > k_0$ and $a \in \mathbb{C}^n$. Therefore, we want $v \in L^2(0, T; \mathbb{C}^m)$ s.t.

$$\begin{cases} \int_0^T (v(T-t), B_{k_0}^* e^{\mathcal{L}_{k_0}^* t} \Phi_0)_{\mathbb{C}^m} dt = \boxed{F(Y_0, \Phi_0)}, & \forall \Phi_0 \in \mathbb{C}^{nk_0}, \\ \int_0^T (v(T-t), B^* e^{(-\lambda_k I_d + A^*)t} a)_{\mathbb{C}^m} dt = \boxed{f_k(y_0, a)}, & \forall a \in \mathbb{C}^n, \forall k > k_0, \end{cases}$$

In some sense, v has to solve an infinite number of null controllability problems for appropriate o.d. systems:

$$\begin{cases} \boxed{Y' = \mathcal{L}_{k_0} Y + B_{k_0} v \text{ on } (0, T), \quad Y(0) = Y_0}; \\ \boxed{Z' = (-\lambda_k I_d + A)Z + Bv \text{ on } (0, T), \quad Z(0) = y_{0k} := (y_0, \phi_k)}, & \forall k > k_0. \end{cases}$$

8. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

From the previous step, we have obtained the **complex matrix exponentials**

$$e^{\mathcal{L}_{k_0}^* t} \quad \text{and} \quad \{e^{(-\lambda_k Id + A^*)t}\}_{k > k_0}.$$

Let us denote $\{\gamma_\ell\}_{1 \leq \ell \leq \tilde{p}} \subset \mathbb{C}$ the set of distinct eigenvalues of $\mathcal{L}_{k_0}^*$ and recall that $\{\mu_i\}_{1 \leq i \leq p} \subset \mathbb{C}$ is the set of distinct eigenvalues of A^* . Then, the set $\Lambda = \{\gamma_\ell\}_{1 \leq \ell \leq \tilde{p}} \cup \{-\lambda_k + \mu_i\}_{k > k_0, 1 \leq i \leq p}$ is the set of eigenvalues of the operator $\partial_{xx} Id + A^*$. Thus, our next purpose is:

Objective

As in the scalar case, construction of a **biorthogonal family** in $L^2(0, T; \mathbb{C})$ to

$$\left\{ t^j e^{\gamma_\ell t}, t^j e^{(-\lambda_k + \mu_i)t} : 1 \leq \ell \leq \tilde{p}, 1 \leq i \leq p, 0 \leq j \leq \eta - 1, k > k_0 \right\},$$

which satisfies appropriate bounds (see (22)). In the previous expression, η is the maximal dimension of the Jordan blocks associated to γ_ℓ and μ_i .

8. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

Let us fix $\eta \geq 1$, an integer, $T \in (0, \infty]$ and $\{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}_+$ a sequence s.t.

$$\Lambda_k \neq \Lambda_j, \quad \forall k, j \geq 1 \text{ with } k \neq j.$$

Let us recall that the family $\{q_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1} \subset L^2(0, T; \mathbb{C})$ is **biorthogonal** to $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$ if one has

$$\int_0^T t^j e^{-\Lambda_k t} q_{l,i}^*(t) dt = \delta_{kl} \delta_{ij}, \quad \forall (k, j), (l, i) : k, l \geq 1, 0 \leq i, j \leq \eta - 1.$$

In addition, we want the family $\{q_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1} \subset L^2(0, T; \mathbb{C})$ to satisfy the property:

For any $\varepsilon > 0$, there is $C(\varepsilon, T) > 0$ s.t. $\|q_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re \Lambda_k}$,
 $\forall k \geq 1$ and $0 \leq j \leq \eta - 1$.

8. The Kalman condition for a class of parabolic systems.

Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

Theorem

Let us fix $T \in (0, \infty]$ and assume that for two positive constants δ and ρ one has

$$\begin{cases} \Re \Lambda_k \geq \delta |\Lambda_k|, & |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1, \\ \sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty. \end{cases}$$

Then, $\exists \{q_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1}$ **biorthogonal** to $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$ such that, for every $\varepsilon > 0$, there exists $C(\varepsilon, T) > 0$ satisfying

$$\|q_{k,j}\|_{L^2(0,T;\mathbb{C})} \leq C(\varepsilon, T) e^{\varepsilon \Re \Lambda_k}, \quad \forall (k,j) : k \geq 1, 0 \leq j \leq \eta - 1.$$

8. The Kalman condition for a class of parabolic systems. Boundary controls

(II) **Biorthogonal families** to appropriate complex matrix exponentials.

Proof:

The proof of this result is very technical. It can be found in
[AMMAR-KHODJA, BENABDALLAH, G.-B., DE TERESA], *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (2011).

8. The Kalman condition for a class of parabolic systems.

Boundary controls

(II.1) Biorthogonal families: EXISTENCE.

Lemma

Assume that $\{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}_+$, with $\Lambda_k \neq \Lambda_j \forall k, j \geq 1$ with $k \neq j$, and

$$\Re \Lambda_k \geq \delta |\Lambda_k| \quad \text{and} \quad \sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

Then, there exists a **biorthogonal family** $\{q_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1} \subset L^2(0, \infty; \mathbb{C})$ to $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$ such that

$$\|q_{k,j}\|_{L^2} \leq C (\Re \Lambda_k)^{\eta(\eta-j)} |1 + \Lambda_k|^{2\eta(\eta-j)} \mathcal{P}_k^{\eta(\eta-j)},$$

with $C = C(\eta) > 0$, a constant, and $\mathcal{P}_k := \prod_{\substack{\ell \geq 1 \\ \ell \neq k}} \left| \frac{1 + \Lambda_k / \Lambda_\ell^*}{1 - \Lambda_k / \Lambda_\ell} \right|$.

8. The Kalman condition for a class of parabolic systems.

Boundary controls

(II.1) Biorthogonal families: EXISTENCE.

Remark

Observe that the assumptions

$$\Re \Lambda_k \geq \delta |\Lambda_k| \quad \text{and} \quad \sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

imply the existence of the **biorthogonal family** $\{q_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1}$ to $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$ in $L^2(0, \infty; \mathbb{C})$. In addition, the norm $\|q_{k,j}\|_{L^2}$ is bound with respect to the Blaschke product

$$\mathcal{P}_k = \prod_{\substack{\ell \geq 1 \\ \ell \neq k}} \left| \frac{1 + \Lambda_k / \Lambda_\ell^*}{1 - \Lambda_k / \Lambda_\ell} \right|.$$

8. The Kalman condition for a class of parabolic systems.

Boundary controls

(II.2) Biorthogonal families: BOUNDS.

Proposition

Let $\{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}_+$ be a sequence satisfying

$$\Re \Lambda_k \geq \delta |\Lambda_k|, \quad |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1, \quad \text{and} \quad \sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty,$$

for $\delta, \rho > 0$. Then, for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\mathcal{P}_k := \prod_{\ell \geq 1, \ell \neq k} \left| \frac{1 + \Lambda_k / \Lambda_\ell^*}{1 - \Lambda_k / \Lambda_\ell} \right| \leq C(\varepsilon) e^{\varepsilon \Re \Lambda_k}, \quad \forall k \geq 1. \quad \blacksquare$$

For a proof of this result: [FATTORINI,RUSSELL] Quart. Appl. Math. (1974/75) (real case) or [FERNÁNDEZ-CARA,G.-B.,DE TERESA], J. Funct. Anal. (2010) (general case).

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Summarizing

For the problem

$$(25) \quad \begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

($A \in \mathcal{L}(\mathbb{C}^n)$ and $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$) we know:

“System (25) is *approximate controllable* at time $T \iff$ System (25) is *null controllable* at time $T \iff$ the *Kalman condition* $\text{rank } \mathcal{K}_k = nk, \quad \forall k \geq 1$ ”.

ESSENTIAL ASSUMPTION: Diffusion matrix $D = I_d$

What happens if $D \neq I_d$???

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Some references

- 1 F. AMMAR-KHODJA, A. BENABDALLAH, M. G-B, L. DE TERESA, *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (9), **96** (2011), no. 6, 555–590.
- 2 G. OLIVE, *Null-controllability for some linear parabolic systems with controls acting on different parts of the domain and its boundary*, Math. Control Signals Systems **23** (2012), no. 4, 257–280.
- 3 A. BENABDALLAH, F. BOYER, M. G-B, G. OLIVE, *Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N -dimensional boundary null controllability in cylindrical domains*, SIAM J. Control Optim. **52** (2014), no. 5, 2970–3001.

9. New phenomena: Minimal time of controllability

9. New phenomena: Minimal time of controllability

We are going to revisit problem (18). With a slightly change of notations, this problem is:

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

where $D = \text{diag}(1, d)$, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. When $d = 1$ (i.e., $D = Id$), we saw

Theorem ($d = 1$)

Let $A_0 \in \mathcal{L}(\mathbb{C}^2)$ and $B \in \mathbb{C}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A_0^* . Then (18) is **approximate and null controllable** at any time $T > 0$ **if and only if** $\text{rank}[A | B] = 2$ and $(\lambda_k = k^2)$

$$\lambda_k - \lambda_j \neq \mu_1 - \mu_2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j. \quad \blacksquare$$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

where $D = \text{diag}(1, d)$, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Theorem ($d \neq 1$)

Under the previous assumptions, system (18) is **approximate controllable** at time $T > 0$ **if and only if** $\sqrt{d} \notin \mathbb{Q}$.

Therefore:

- 1 If $d = 1$, (18) is **approximate and null controllable** at any $T > 0$.
- 2 If $d \neq 1$, we only know that system (18) is **approximate controllable** at time $T > 0$ **if and only if** $\sqrt{d} \notin \mathbb{Q}$.

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

where $D = \text{diag}(1, d)$, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Assumption

In the sequel, $D = \text{diag}(1, d)$ with $d \neq 1$ and $\sqrt{d} \notin \mathbb{Q}$.

Goal

Analyze the **null controllability** properties at time $T > 0$ of system (18).

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Let φ be a solution of the adjoint problem:

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^* \varphi = 0 & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 \in H_0^1(0, \pi)^2 & \text{in } (0, \pi). \end{cases}$$

If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0 \rangle - \langle y_0, \varphi(0) \rangle = \int_0^T v(t) B^* D \varphi_x(0, t) dt$$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Let φ be a solution of the adjoint problem:

$$\begin{cases} -\varphi_t - D\varphi_{xx} + A_0^* \varphi = 0 & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 \in H_0^1(0, \pi)^2 & \text{in } (0, \pi). \end{cases}$$

If y is a solution of the direct problem, then

$$\langle y(T), \varphi_0 \rangle - \langle y_0, \varphi(0) \rangle = \int_0^T v(t) B^* D \varphi_x(0, t) dt$$

Thus $y(T) = 0 \iff \exists v \in L^2(0, T)$ such that

$$\int_0^T v(t) B^* D \varphi_x(0, t) dt = -\langle y_0, \varphi(0) \rangle, \quad \forall \varphi_0 \in H_0^1(0, \pi; \mathbb{R}^2)$$

9. New phenomena: Minimal time of controllability

Fattorini-Russell Method

9. New phenomena: Minimal time of controllability

Fattorini-Russell Method

- $\sigma(-D\partial_{xx}^2 + A_0^*) = \bigcup_{k \geq 1} \{k^2, dk^2\} := \bigcup_{k \geq 1} \{\lambda_{k,1}, \lambda_{k,2}\}$.
- $\{\Phi_{k,i}\}$ a (Riesz) basis of $H_0^1(0, \pi)^2$, where $\Phi_{k,i} = V_{k,i} \sin kx$, $i = 1, 2$ are eigenfunctions of the operator $-D\partial_{xx}^2 + A_0^*$.
- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $k^2 D + A_0^*$ associated to the eigenvalues k^2, dk^2 .

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Objective: Existence of $v \in L^2(0, T)$ s.t.

$$\int_0^T v(t) B^* D \varphi_x(0, t) dt = - \langle y_0, \varphi(0) \rangle, \quad \forall \varphi_0 \in H_0^1(0, \pi; \mathbb{R}^2)$$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Objective: Existence of $v \in L^2(0, T)$ s.t.

$$\int_0^T v(t) B^* D \varphi_x(0, t) dt = -\langle y_0, \varphi(0) \rangle, \quad \forall \varphi_0 \in H_0^1(0, \pi; \mathbb{R}^2)$$

- Choosing $\varphi_0 = \Phi_{k,i}$, we have $\varphi(\cdot, t) = e^{-\lambda_{k,i}(T-t)} \Phi_{k,i}$ and

$$\varphi(x, 0) = e^{-\lambda_{k,i}T} \Phi_{k,i}(x), \quad \varphi_x(0, t) = ke^{-\lambda_{k,i}(T-t)} V_{k,i}$$

- The identity connecting y and φ writes (**moment problem**)

$$kB^* D V_{k,i} \int_0^T v(T-t) e^{-\lambda_{k,i}t} dt = -e^{-\lambda_{k,i}T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall (k, i)$$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Approximate controllability: a necessary condition (I)

- $kB^*DV_{k,i} \int_0^T v(T-t)e^{-\lambda_{k,i}t} dt = -e^{-\lambda_{k,i}T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall (k, i)$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Approximate controllability: a necessary condition (I)

- $kB^*DV_{k,i} \int_0^T v(T-t)e^{-\lambda_{k,i}t} dt = -e^{-\lambda_{k,i}T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall(k, i)$
- A necessary condition: $B^*DV_{k,i} \neq 0$ for all $k \geq 1, i = 1, 2$
- Recall $d \neq 1$,

$$B^* = (0, 1), \quad V_{k,1} = \begin{pmatrix} 1 \\ \frac{1}{(d-1)k^2} \end{pmatrix}, \quad V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

So, here $B^*DV_{k,i} \neq 0, \quad \forall k \geq 1, i = 1, 2$ (**algebraic Kalman condition**)

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Approximate controllability: a necessary condition (II)

$$\lambda_{k,1} = \lambda_{j,2} = \lambda \Rightarrow \begin{cases} kB^* DV_{k,1} \int_0^T v(T-t)e^{-\lambda t} dt = -e^{-\lambda T} \langle y_0, \Phi_{k,1} \rangle \\ jB^* DV_{j,2} \int_0^T v(T-t)e^{-\lambda t} dt = -e^{-\lambda T} \langle y_0, \Phi_{j,2} \rangle \end{cases}$$

So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}}$$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

Approximate controllability: a necessary condition (II)

$$\lambda_{k,1} = \lambda_{j,2} = \lambda \Rightarrow \begin{cases} kB^*DV_{k,1} \int_0^T v(T-t)e^{-\lambda t} dt = -e^{-\lambda T} \langle y_0, \Phi_{k,1} \rangle \\ jB^*DV_{j,2} \int_0^T v(T-t)e^{-\lambda t} dt = -e^{-\lambda T} \langle y_0, \Phi_{j,2} \rangle \end{cases}$$

So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$. This leads to

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}}$$

In the sequel, we will assume $\sqrt{d} \notin \mathbb{Q}$, i.e., the eigenvalues of $-D\partial_{xx}^2 + A_0^*$ with Dirichlet boundary conditions are pairwise distinct.

9. New phenomena: Minimal time of controllability

(18)

$$\begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$$kB^*DV_{k,i} \int_0^T v(T-t)e^{-\lambda_{k,i}t} dt = -e^{-\lambda_{k,i}T} \langle y_0, \Phi_{k,i} \rangle, \quad \forall (k, i)$$

Summarizing

Let $m_{k,i} = -\langle y_0, \Phi_{k,i} \rangle$, $b_{k,i} = kB^*DV_{k,i}$ (for any $\varepsilon > 0$, $|m_{k,i}| \leq C_\varepsilon e^{\varepsilon\lambda_{k,i}}$ and

$$|b_{k,i}| \geq C_\varepsilon e^{-\varepsilon\lambda_{k,i}}),$$

$$\exists? v \in L^2(0, T) : \int_0^T v(T-t)e^{-\lambda_{k,i}t} dt = \frac{m_{k,i}}{b_{k,i}} e^{-\lambda_{k,i}T}, \quad \forall k \geq 1, i = 1, 2$$

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

Let $\Lambda = \{\Lambda_k\}_{k \geq 1} \subset (0, \infty)$ be a sequence with **pairwise distinct elements**:

$$\sum_{k \geq 1} \frac{1}{\Lambda_k} < \infty$$

Goal: Given $\{m_k\}_{k \geq 1}, \{b_k\}_{k \geq 1} \subset \mathbb{R}$ satisfying $|m_k| \leq C_\varepsilon e^{\varepsilon \Lambda_k}$ and

$|b_k| \geq C_\varepsilon e^{-\varepsilon \Lambda_k}$, find $v \in L^2(0, T)$ s.t.

$$\int_0^T v(T-t) e^{-\Lambda_k t} dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k \geq 1.$$

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

Recall that the assumption

$$\sum_{k \geq 1} \frac{1}{\Lambda_k} < \infty$$

implies:

Theorem

Under the previous assumptions, $\{e^{-\Lambda_k t}\}_{k \geq 1} \subset L^2(0, T)$ admits a **biorthogonal family** $\{q_k\}_{k \geq 1}$ in $L^2(0, T)$, i.e.:

$$\int_0^T e^{-\Lambda_k t} q_l(t) dt = \delta_{kl}, \quad \forall k, l \geq 1$$

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

A formal solution to

$$\int_0^T v(T-t)e^{-\Lambda_k t} dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k \geq 1,$$

is v given by:
$$v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t),$$

Question: $v \in L^2(0, T)$?, i.e., is the series $\sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$ convergent in $L^2(0, T)$?

But this question itself amounts to:

$$\|q_k\|_{L^2(0, T)} \underset{k \rightarrow \infty}{\sim} ?$$

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

Theorem

Assume that $\sum_{k \geq 1} \frac{1}{\Lambda_k} < \infty$ and (**gap condition**)

$$\boxed{\exists \rho > 0 : |\Lambda_k - \Lambda_j| \geq \rho |k - j|, \quad \forall k, j.}$$

Then, for any $\varepsilon > 0$ one has

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon \Lambda_k}, \quad \forall k \geq 1,$$

and, for $T > 0$, the control $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0, T)$.

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

Theorem

Assume that $\sum_{k \geq 1} \frac{1}{\Lambda_k} < \infty$ and (**gap condition**)

$$\boxed{\exists \rho > 0 : |\Lambda_k - \Lambda_j| \geq \rho |k - j|, \quad \forall k, j.}$$

Then, for any $\varepsilon > 0$ one has

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon \Lambda_k}, \quad \forall k \geq 1,$$

and, for $T > 0$, the control $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0, T)$.

Recall that in our case $\Lambda = \{\Lambda_k\}_{k \geq 1} = \{j^2, dj^2\}_{j \geq 1}$, and the property

$$\boxed{\exists \rho > 0 : |\Lambda_k - \Lambda_j| \geq \rho |k - j|, \quad \forall k, j,}$$

does not hold.

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

How does this fact affect our problem??

Theorem

Assume $\sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty$. Then, for any $\varepsilon > 0$ one has

$$C_{1,\varepsilon} \frac{e^{-\varepsilon \Lambda_k}}{|W'(\Lambda_k)|} \leq \|q_k\|_{L^2(0,T)} \leq C_{2,\varepsilon} \frac{e^{\varepsilon \Lambda_k}}{|W'(\Lambda_k)|}, \quad \forall k \geq 1,$$

where $W(z)$ is the Blaschke product:

$$W(z) = \prod_{k=1}^{\infty} \frac{1 - z/\Lambda_k}{1 + z/\Lambda_k},$$

$$W'(\Lambda_k) = -\frac{1}{2\Lambda_k} \prod_{j \neq k} \frac{1 - \Lambda_k/\Lambda_j}{1 + \Lambda_k/\Lambda_j}$$

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

Definition

The **condensation index** of $\Lambda = \{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}$ is:

$$c(\Lambda) = \limsup_{k \rightarrow \infty} \frac{-\log |W'(\Lambda_k)|}{\Re(\Lambda_k)} \in [0, +\infty].$$

Corollary

For any $\varepsilon > 0$ one has

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\Lambda_k}, \quad \forall k \geq 1.$$

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

Recall that we had m_k s.t. $|m_k| \leq C_\varepsilon e^{\varepsilon \Lambda_k}$, $|b_k| \geq C_\varepsilon e^{-\varepsilon \Lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0, T)$ and

$$\int_0^T v(T-t) e^{-\Lambda_k t} dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k,$$

We took $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$.

9. New phenomena: Minimal time of controllability

The moment problem: Abstract setting

Recall that we had m_k s.t. $|m_k| \leq C_\varepsilon e^{\varepsilon\Lambda_k}$, $|b_k| \geq C_\varepsilon e^{-\varepsilon\Lambda_k}$, for any $\varepsilon > 0$, and we wanted to solve: $v \in L^2(0, T)$ and

$$\int_0^T v(T-t)e^{-\Lambda_k t} dt = \frac{m_k}{b_k} e^{-\Lambda_k T}, \quad \forall k,$$

We took $v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t)$.

From the previous result: Given $\varepsilon > 0$:

$$\left| \frac{m_k}{b_k} \right| e^{-\Lambda_k T} \|q_k\|_{L^2(0, T)} \leq C_\varepsilon e^{-\Lambda_k(T-c(\Lambda)-\varepsilon)}$$

Then

$$T > c(\Lambda) \implies v(T-t) = \sum_{k \geq 1} \frac{m_k}{b_k} e^{-\Lambda_k T} q_k(t) \in L^2(0, T).$$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

In our case,

$$\Lambda_d := \{\Lambda_k\}_{k \geq 1} = \{j^2, dj^2\}_{j \geq 1}.$$

Then

If $T > c(\Lambda_d)$, system (18) is null controllable at time T , where $c(\Lambda_d)$ is the **condensation index** of the sequence Λ_d .

9. New phenomena: Minimal time of controllability

Index of condensation: Some background

$$(18) \quad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

- The **index of condensation** of a sequence $\Lambda = \{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}$ is a real number $c(\Lambda) \in [0, +\infty]$ associated with this sequence and which “measures” the condensation at infinity.

$$c(\Lambda) = \limsup_{k \rightarrow \infty} \frac{-\log |W'(\Lambda_k)|}{\Re(\Lambda_k)} \in [0, +\infty],$$

$$W'(\Lambda_k) = \frac{-1}{2\Lambda_k} \prod_{j \neq k}^{\infty} \frac{1 - \frac{\Lambda_k}{\Lambda_j}}{1 + \frac{\Lambda_k}{\Lambda_j}}.$$

- This notion has been :
 - introduced by V.I. Bernstein in 1933:
[Leçons sur les progrès récents de la théorie des séries de Dirichlet](#)
for real sequences,
 - extended by J. R. Shackell in 1967 for complex sequences.

9. New phenomena: Minimal time of controllability

Index of condensation: Some examples

① **Gap property:** $\exists \rho > 0 : |\Lambda_k - \Lambda_l| \geq \rho |k - l| \Rightarrow c(\Lambda) = 0$.

In particular: for the scalar Dirichlet-Laplacien operator: $\Lambda_k = k^2$,
 $|\Lambda_k - \Lambda_l| = |k^2 - l^2| \geq |k - l|$. So

$$\Lambda = \{k^2\}_{k \geq 1} \Rightarrow c(\Lambda) = 0.$$

9. New phenomena: Minimal time of controllability

Index of condensation: Some examples

- ① **Gap property:** $\exists \rho > 0 : |\Lambda_k - \Lambda_l| \geq \rho |k - l| \Rightarrow c(\Lambda) = 0$.

In particular: for the scalar Dirichlet-Laplacien operator: $\Lambda_k = k^2$,
 $|\Lambda_k - \Lambda_l| = |k^2 - l^2| \geq |k - l|$. So

$$\Lambda = \{k^2\}_{k \geq 1} \Rightarrow c(\Lambda) = 0.$$

- ② $\alpha > 1, \beta > 0$ and $\Lambda = \{\Lambda_k\}_{k \geq 1}$ with $\Lambda_{2k} = k^\alpha, \Lambda_{2k+1} = k^\alpha + e^{-k^\beta}$

$$c(\Lambda) = \begin{cases} 0 & \beta < \alpha \\ 1 & \beta = \alpha \\ +\infty & \beta > \alpha \end{cases} \quad (\text{Note that } \liminf |\Lambda_{k+1} - \Lambda_k| = 0)$$

9. New phenomena: Minimal time of controllability

Index of condensation: Some examples

- ① **Gap property:** $\exists \rho > 0 : |\Lambda_k - \Lambda_l| \geq \rho |k - l| \Rightarrow c(\Lambda) = 0$.

In particular: for the scalar Dirichlet-Laplacian operator: $\Lambda_k = k^2$,
 $|\Lambda_k - \Lambda_l| = |k^2 - l^2| \geq |k - l|$. So

$$\Lambda = \{k^2\}_{k \geq 1} \Rightarrow c(\Lambda) = 0.$$

- ② $\alpha > 1, \beta > 0$ and $\Lambda = \{\Lambda_k\}_{k \geq 1}$ with $\Lambda_{2k} = k^\alpha, \Lambda_{2k+1} = k^\alpha + e^{-k^\beta}$

$$c(\Lambda) = \begin{cases} 0 & \beta < \alpha \\ 1 & \beta = \alpha \\ +\infty & \beta > \alpha \end{cases} \quad (\text{Note that } \liminf |\Lambda_{k+1} - \Lambda_k| = 0)$$

- ③ $\Lambda = \{\Lambda_k\}_{k \geq 1}$ with

$$\Lambda_{k^2+n} = k^2 + ne^{-k^2}, \quad n \in \{0, \dots, 2k\}, \quad k \geq 1$$

$$c(\Lambda) = +\infty$$

9. New phenomena: Minimal time of controllability

The controllability result

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$$D = \text{diag}(1, d), \quad \Lambda_d = \{k^2, dk^2\}_{k \geq 1}, \quad \sqrt{d} \notin \mathbb{Q}.$$

We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ such that if $T > T_0$ then system (18) is null controllable at time T

9. New phenomena: Minimal time of controllability

The controllability result

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$$D = \text{diag}(1, d), \quad \Lambda_d = \{k^2, dk^2\}_{k \geq 1}, \quad \sqrt{d} \notin \mathbb{Q}.$$

We have proved:

Theorem

There exists $T_0 = c(\Lambda_d) \in [0, +\infty]$ such that if $T > T_0$ then system (18) is null controllable at time T

$T > c(\Lambda_d)$ is a sufficient condition for the null controllability of system (18) at time T . But,

what happens if $T < c(\Lambda_d)$?

9. New phenomena: Minimal time of controllability

The non-controllability result

One can prove:

Theorem

Let us take

$$T_0 = c(\Lambda_d) \in [0, +\infty].$$

Then, if $T < T_0$, system (18) is not null controllable at time T .

Idea of the proof

By contradiction:

- The null controllability at time T is equivalent to: $\exists C_T > 0$ s.t.

$$\sum_{n,i} e^{-2\Lambda_{n,i}T} |a_{n,i}|^2 \leq C_T \int_0^T \left| \sum_{n,i} nB^* DV_{n,i} e^{-\Lambda_{n,i}t} a_{n,i} \right|^2 dt, \quad \forall \{a_{n,i}\}_{n,i} \in \ell^2.$$

- Argument: Use the overconvergence of Dirichlet series.

9. New phenomena: Minimal time of controllability

The controllability result

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

The controllability result

- ① $\forall T > 0$: **Approximate controllability** at time T if and only if

$$\sqrt{d} \notin \mathbb{Q}.$$

- ② Assume $\sqrt{d} \notin \mathbb{Q}$, $\exists T_0 = c(\Lambda_d) \in [0, +\infty]$ such that

① the system is null controllable at time T if $T > T_0$

② Even if $\sqrt{d} \notin \mathbb{Q}$, if $T < T_0$ the system is **not null controllable** at time T !

9. New phenomena: Minimal time of controllability

The controllability result

$$(18) \quad \boxed{\begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}}$$

In fact, the good minimal time is

$$T_0 = \limsup_{k \rightarrow \infty} \frac{-(\log |b_k| + \log |W'(\Lambda_k)|)}{\Re(\Lambda_k)} \in [0, \infty]$$

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0 ? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k \geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$T_0 > 0$?

Is it possible to have a minimal time of control > 0 ? I.e., for $\Lambda_d = \{k^2, dk^2\}_{k \geq 1}$ with $\sqrt{d} \notin \mathbb{Q}$, is it possible that $c(\Lambda_d) > 0$?

Theorem

For any $\tau \in [0, +\infty]$, there exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = \tau$.

Remark

- There exists $\sqrt{d} \notin \mathbb{Q}$ such that $c(\Lambda_d) = +\infty$ (LUCA, DE TERESA).
- $c(\Lambda_d) = 0$ for almost $d \in (0, \infty)$ such that $\sqrt{d} \notin \mathbb{Q}$.
- For any $\tau \in [0, +\infty]$, the set $\{d \in (0, \infty) : c(\Lambda_d) = \tau\}$ is dense in $(0, +\infty)$.

9. New phenomena: Minimal time of controllability

$$(18) \quad \begin{cases} y_t - Dy_{xx} + A_0y = 0 & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$$\text{where } D = \text{diag}(1, d), \quad A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Third phenomenon

For system (18): If $\sqrt{d} \notin \mathbb{Q}$, then,

- 1 **Approximate controllability:** System (18) is approximately controllable at any time $T > 0$.
- 2 **Null controllability:** System (18) is null controllable is $T > T_0 = c(\Lambda_d)$ and is not if $T < T_0 = c(\Lambda_d)$.

9. New phenomena: Minimal time of controllability

Remark

This minimal time also arises in other parabolic problems (degenerated problems):

BEAUCHARD, CANNARSA, GUGLIELMI, *Null controllability of Grushin-type operators in dimension two. J. Eur. Math. Soc. (JEMS) (2014).*

BEAUCHARD, MILLER, MORANCEY, *2d Grushin-type equations: Minimal time and null controllable data, J. Differential Equations 259 (2015), no. 11*

Reference

F. AMMAR KHODJA, A. BENABDALLAH, M.G.-B., L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences, J. Funct. Anal. 267 (2014).*

<http://personal.us.es/manoloburgos>

10. New phenomena: Dependence on the position of the control set

10. New phenomena: Geometrical dependence

Let us fix $T > 0$ and $\omega = (a, b) \subset (0, \pi)$. We consider the coupled parabolic systems:

$$(28) \quad \left\{ \begin{array}{ll} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T := (0, \pi) \times (0, T), \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{array} \right.$$

In (28), 1_ω is the characteristic function of the set ω , $y(x, t)$ is the state, $y_0 \in L^2(0, \pi; \mathbb{R}^2)$ is the **initial datum** and

- $q \in L^\infty(0, \pi)$ is a given function, $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2)$ is a constant matrix and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a constant vector of \mathbb{R}^2 ;
- $u \in L^2(Q_T)$ is a scalar control function.

10. New phenomena: Geometrical dependence

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Remark

If $q \in L^\infty(0, \pi)$ satisfies: There exist an open subset $\omega_0 \subseteq \omega$ and a constant $\delta > 0$ s.t.

$$\boxed{q \geq \delta > 0 \text{ a.e. } \omega_0} \quad \text{or} \quad \boxed{q \leq -\delta < 0 \text{ a.e. } \omega_0}$$

$\left(\implies \boxed{\text{Supp } q \cap \omega \neq \emptyset} \right)$, then it is possible to repeat the arguments of section 2 and prove:

Theorem

Under the previous assumption, system (28) is approximately and exactly controllable to zero at any time $T > 0$.

10. New phenomena: Geometrical dependence

Let us consider the 2×2 linear reaction-diffusion system

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

where $q \in L^\infty(Q_T)$, $y_0 \in L^2(0, \pi; \mathbb{R}^2)$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$\omega = (a, b) \subset (0, \pi)$ and $u \in L^2(Q_T)$ is a scalar control function.

10. New phenomena: Geometrical dependence

Let us consider the 2×2 linear reaction-diffusion system

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

where $q \in L^\infty(Q_T)$, $y_0 \in L^2(0, \pi; \mathbb{R}^2)$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$\omega = (a, b) \subset (0, \pi)$ and $u \in L^2(Q_T)$ is a scalar control function.

No sign conditions on q .

$$\omega \cap \text{Supp } q = \emptyset$$

10. New phenomena: Geometrical dependence

(28)

$$\begin{cases} y_t - y_{xx} + q(x)y = 0 & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Theorem (Ammar Khodja, Benabdallah, G-B, de Teresa (2011))

Assume $I_k(q) \neq 0$ for any $k \geq 1$, where

$$(29) \quad I_k(q) := \int_0^\pi q(x) |\sin(kx)|^2 dx,$$

and

$$\int_0^\pi q(x) dx \neq 0.$$

Then, for any $T > 0$, system (28) is **null controllable** at time T .

10. New phenomena: Geometrical dependence

(28)

$$\begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Null controllability properties of system (28) when

$$\int_0^\pi q(x) dx = 0?$$

In order to simplify the problem, we will assume the **geometrical assumption**:

Assumption (A1)

The function q satisfies $\text{Supp } q \subset [0, a]$ or $\text{Supp } q \subset [b, \pi]$ ($\omega = (a, b)$).

10. New phenomena: Geometrical dependence

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Proposition (Boyer and Olive (2014))

*Under the geometrical assumption (A1), system (28) is **approximately controllable** at time $T > 0$ if and only if*

$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

10. New phenomena: Geometrical dependence

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Proposition (Boyer and Olive (2014))

Under the geometrical assumption (A1), system (28) is **approximately controllable** at time $T > 0$ if and only if

$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

Remarks

- 1 The approximate controllability of system (28) does not depend on T .
- 2 Again, condition

$$I_k(q) \neq 0, \quad \forall k \geq 1.$$

is necessary for the null controllability of system (28) at time $T > 0$

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

We have a Riesz basis $\mathcal{B} := \left\{ \Phi_{k,1}^*, \Phi_{k,2}^* \right\}_{k \geq 1}$ of eigenfunctions and generalized eigenfunctions of the operator $L^* := -\frac{d^2}{dx^2} + q(x)A_0^*$ associated to the eigenvalue k^2 (**simple**).

Idea:

We will work with controls $u(x, t) = f(x)v(t)$ with $v \in L^2(0, T)$ and $f \in L^2(0, \pi)$ (appropriate) satisfies $\text{Supp } f \subset \omega$.

Objective

Apply Fattorini-Russell method: **moment problem**

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

The moment problem

Find $v \in L^2(0, T)$ s.t.

$$\begin{cases} \int_0^T v(T-t) \boxed{e^{-k^2t}} dt = \frac{m_{k,1}}{f_k} e^{-k^2T}, \quad \forall k \geq 1, \\ \int_0^T v(T-t) \boxed{te^{-k^2t}} dt = \frac{m_{k,2}}{I_k(q)f_k} e^{-k^2T}, \quad \forall k \geq 1, \end{cases}$$

where $\boxed{|m_{k,i}| \leq C_\varepsilon e^{\varepsilon\lambda_k}}$ and $\boxed{|f_k| \sim k^{-3} \geq C_\varepsilon e^{-\varepsilon\lambda_k}}$ ($i = 1, 2$).

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

The moment problem

Find $v \in L^2(0, T)$ s.t.

$$\begin{cases} \int_0^T v(T-t)e^{-k^2 t} dt = \frac{m_{k,1}}{f_k} e^{-k^2 T}, \quad \forall k \geq 1, \\ \int_0^T v(T-t)te^{-k^2 t} dt = \frac{m_{k,2}}{I_k(q) f_k} e^{-k^2 T}, \quad \forall k \geq 1, \end{cases}$$

where $|m_{k,i}| \leq C_\varepsilon e^{\varepsilon \lambda_k}$ and $|f_k| \sim k^{-3} \geq C_\varepsilon e^{-\varepsilon \lambda_k}$ ($i = 1, 2$).

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Conclusion

We can obtain the positive controllability result if

$$T > \tilde{T}_0(q) = \limsup \frac{-\log |I_k(q)|}{k^2},$$

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$. Then, if $T > \tilde{T}_0(q)$, system (28) is null-controllable at time T .

Does the minimal time depend on the choice $u(x, t) = f(x)v(t)$?

What happens if $T < \tilde{T}_0(q)$?

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

As before, the null controllability property for system (28) is equivalent to the **observability inequality**:

$$\|\varphi(\cdot, 0)\|_{(L^2)^2}^2 \leq C_T \int_0^T \int_\omega |\varphi_2(x, t)|^2 dx dt,$$

for the solutions to **the adjoint problem**

$$\begin{cases} -\varphi_t - \varphi_{xx} + q(x)A_0^* \varphi = 0 & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \end{cases}$$

10. New phenomena: Geometrical dependence

Null controllability

$$\|\varphi(\cdot, 0)\|_{(L^2)^2}^2 \leq C_T \int_0^T \int_{\omega} |\varphi_2(x, t)|^2 dx dt,$$

If $T < \tilde{T}_0(q)$, we can prove that the inequality does not hold **reasoning by contradiction**: Then system

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_{\omega} & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

is not null controllable at time T .

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu \mathbf{1}_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

$$\omega \cap \text{Supp } q = \emptyset$$

Theorem

Assume $I_k(q) \neq 0$ for all $k \geq 1$ and let:

$$\tilde{T}_0(q) := \limsup \frac{-\log |I_k(q)|}{k^2} \in [0, +\infty]$$

Then,

- 1 If $T > \tilde{T}_0(q)$, then system (28) is null-controllable at time T .
- 2 If $\text{Supp } q \subset [0, a]$ or $\text{Supp } q \subset [b, \pi]$, for any $T < \tilde{T}_0(q)$, the system is not null-controllable at time T .

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Remarks

- 1 The previous results cannot be obtained using Carleman inequalities.
- 2 Due to the geometrical assumption

The function q satisfies $\text{Supp } q \subset [0, a]$ or $\text{Supp } q \subset [b, \pi]$ ($\omega = (a, b)$)
the boundary and distributed null controllability results coincide.

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

General case

$$\omega = (a, b) \subset (0, \pi) \text{ and } \text{Supp } q \cap \omega = \emptyset.$$

The condition $I_k(q) \neq 0$ is no longer necessary:

$$I_{1,k}(q) := \int_0^a q(x) |\sin(kx)|^2 dx; \quad I_{2,k}(q) := \int_b^1 q(x) |\sin(kx)|^2 dx$$

$$I_k(q) = I_{1,k}(q) + I_{2,k}(q) = \int_0^\pi q(x) |\sin(kx)|^2 dx;$$

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \left\{ \begin{array}{ll} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{array} \right.$$

Proposition (Boyer and Olive (2014))

If $\omega = (a, b)$, system (28) is **approximately controllable** at time $T > 0$ if and only if

$$|I_k(q)| + |I_{1,k}(q)| \neq 0, \quad \forall k \geq 1.$$

The proof uses the independence of the functions $\sin(kx)$ and $\cos(kx)$ in ω .

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Remarks

- 1 The approximate controllability of system (28) does not depend on T .
- 2 Again, condition

$$|I_k(q)| + |I_{1,k}(q)| \neq 0, \quad \forall k \geq 1.$$

is necessary for the null controllability of system (28) at time $T > 0$.

Null controllability of system (28)???

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu_1 \omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

In this case we can have $I_k(q) = 0$, and then,

$$L := -\frac{d^2}{dx^2} + q(x)A_0 : L^2(0, \pi; \mathbb{R}^2) \longrightarrow L^2(0, \pi; \mathbb{R}^2)$$

has eigenvalues (k^2) of multiplicity 2.

Idea

Apply Fattorini-Russell's method with control under the form:

$$u(x, t) = f_1(x)v_1(t) + f_2(t)v_2(t)$$

with $\text{Supp } f_1, \text{Supp } f_2 \subset (a, b)$

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Theorem

Let $\omega = (a, b) \subset (0, \pi)$ and $q \in L^\infty(Q_T)$ satisfying $\omega \cap \text{Supp } q = \emptyset$,

$$|I_{1,k}(q)|^2 + |I_{2,k}(q)|^2 \neq 0 \quad (\iff |I_{1,k}(q)|^2 + |I_k(q)|^2 \neq 0).$$

and

$$T_0(q) = \limsup \frac{\min [-\log |I_{1,k}(q)|, -\log |I_k(q)|]}{k^2}$$

Then,

- 1 If $T > T_0(q)$, then system (28) is null-controllable at time T .
- 2 For any $T < T_0(q)$, the system is not null-controllable at time T .

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Remark

If

$$|I_{1,k}(q)|^2 + |I_{2,k}(q)|^2 \neq 0$$

and

$$\int_0^a q(x) dx \neq 0 \quad \text{or} \quad \int_b^\pi q(x) dx \neq 0 \quad \text{or} \quad \int_0^\pi q(x) dx \neq 0,$$

Then $T_0(q) = 0$ (**Null controllability** of system (28) for every $T > 0$).

10. New phenomena: Geometrical dependence

Null controllability

Idea of the proof:

- 1 The reasoning for $T < T_0(q)$ is by contradiction.
- 2 For proving the positive controllability result for $T > T_0(q)$ we have to "measure" the linear independence of $B^* \Phi_{k,1}^* := \psi_k$ and

$B^* \Phi_{k,2}^* := \sin(kx)$ in ω ($\Phi_{k,1}^*$ and $\Phi_{k,2}^*$ are the eigenfunctions or the eigenfunction and the generalized eigenfunction of $L^* := -\frac{d^2}{dx^2} + q(x)A_0^*$ associated to k^2). Thanks to the assumption $\omega \cap \text{Supp } q = \emptyset$ and the expression of ψ_k in ω this amounts to prove

$$\det \begin{pmatrix} f_{1,k} & f_{2,k} \\ \tilde{f}_{1,k} & \tilde{f}_{2,k} \end{pmatrix} \geq \frac{C}{k^m} \frac{I_{1,k}(q)}{I_k(q)}, \text{ when } I_{1,k}(q) \neq 0 \text{ and } I_k(q) \neq 0$$

where $C > 0$, $m \geq 1$, $f_{i,k}$ is the Fourier coefficient of f_i and

$$\tilde{f}_{i,k} = \int_{\omega} f_i(x) \psi_k(x) dx, \quad k \geq 1, \quad i = 1, 2.$$

10. New phenomena: Geometrical dependence

Null controllability

Example

$$q(x) = \begin{cases} 1 & \text{si } x \in (a_1, a_1 + \ell) \\ -1 & \text{si } x \in (a_2, a_2 + \ell), \end{cases}$$

$a_1 > 0$, $a_1 + \ell < a_2$, $a_2 + \ell < \pi$, $\ell > 0$ and $\omega = (a, b)$.

- ① $\omega \cap \text{Supp } q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability**
 $\forall T > 0$.

10. New phenomena: Geometrical dependence

Null controllability

Example

$$q(x) = \begin{cases} 1 & \text{si } x \in (a_1, a_1 + \ell) \\ -1 & \text{si } x \in (a_2, a_2 + \ell), \end{cases}$$

$a_1 > 0$, $a_1 + \ell < a_2$, $a_2 + \ell < \pi$, $\ell > 0$ and $\omega = (a, b)$.

① $\omega \cap \text{Supp } q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability**
 $\forall T > 0$.

② $\omega = (a, b) \subseteq (0, a_1)$: $I_{1,k}(q) = \int_0^a q(x) dx = 0$, $\forall k$,

$$I_{2,k}(q) = -\frac{2}{k\pi} \sin(k(a_1 + a_2 + \ell)) \sin(k(a_2 - a_1)) \sin(k\ell)$$

10. New phenomena: Geometrical dependence

Null controllability

Example

$$q(x) = \begin{cases} 1 & \text{si } x \in (a_1, a_1 + \ell) \\ -1 & \text{si } x \in (a_2, a_2 + \ell), \end{cases}$$

$a_1 > 0, a_1 + \ell < a_2, a_2 + \ell < \pi, \ell > 0$ and $\omega = (a, b)$.

① $\omega \cap \text{Supp } q \neq \emptyset$ or $\omega \subseteq (a_1 + \ell, a_2)$: $T_0(q) = 0$. **Null controllability**
 $\forall T > 0$.

② $\omega = (a, b) \subseteq (0, a_1)$: $I_{1,k}(q) = \int_0^a q(x) dx = 0, \forall k$,

$$I_{2,k}(q) = -\frac{2}{k\pi} \sin(k(a_1 + a_2 + \ell)) \sin(k(a_2 - a_1)) \sin(k\ell)$$

- **Aprox. Contr.** $T > 0 \iff \boxed{(a_1 + a_2 + \ell)/\pi}, \boxed{(a_2 - a_1)/\pi}, \boxed{\ell/\pi} \notin \mathbb{Q}$.
- Given $\tau \in [0, \infty]$, $\exists a_1, a_2$ y ℓ satisfying the previous property s.t.

$\boxed{T_0(q) = \tau}$. **Minimal time** of null controllability which could be

$\boxed{T_0(q) = \infty}$.

10. New phenomena: Geometrical dependence

Null controllability

$$(28) \quad \begin{cases} y_t - y_{xx} + q(x)A_0 y = Bu1_\omega & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases}$$

Fourth phenomenon

For system (28): $\omega = (a, b) \subset (0, \pi)$ and $\omega \cap \text{Supp } q = \emptyset$, then,

- 1 The **approximate controllability** is not equivalent to the **null controllability**.
- 2 **Null controllability**: The controllability result depends on the relative position of ω with respect to $\text{Supp } q$.

Summarizing

Scalar case versus systems (parabolic problems)

	SCALAR CASE	SYSTEMS
boundary \Leftrightarrow distributed control	Yes	No
approximate \Leftrightarrow null controllability	Yes	No
minimal time for controlling	No	Yes
geometrical conditions	No	Yes

Thank you for your attention!!

8. The Kalman condition for a class of parabolic systems. Boundary controls

8. The Kalman condition for a class of parabolic systems. Boundary controls

8. The Kalman condition for a class of parabolic systems.

Boundary controls

Asqui Let us now revisit the boundary controllability problem:

$$(18) \quad \begin{cases} y_t - D y_{xx} = A y & \text{in } Q_T, \\ y|_{x=0} = B v, \quad y|_{x=1} = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0 & \text{in } (0, \pi), \end{cases}$$

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad d_1 \neq d_2, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We know:

- 1 **Approximate controllability:** System (18) is approximately controllable at time $T \iff \sqrt{d_1/d_2} \notin \mathbb{Q} \iff$ the eigenvalues of $\mathcal{R} = D\partial_{xx} + A^*$ are **simple**.
- 2 **Null controllability:** There are d_1, d_2 s.t. $\sqrt{d_1/d_2} \notin \mathbb{Q}$ and System (18) is not null controllable at any time $T > 0$.

8. The Kalman condition for a class of parabolic systems. Boundary controls

Let us analyze a little more the null controllability problem for System (18) when $\nu = \sqrt{d_1/d_2} \notin \mathbb{Q}$. It is possible to apply the **moment method** to (18) (now, much simpler) and reduce the null controllability problem to the **existence** (with **bounds**) of a biorthogonal family to appropriate exponentials.

Spectrum of $-\mathcal{R}^*$

The operator $-\mathcal{R}^* = -D\partial_{xx} - A^*$ has a sequence of positive real eigenvalues

$$\left\{ \frac{\ell^2}{d_1} \right\}_{\ell \geq 1} \cup \left\{ \frac{i^2}{d_2} \right\}_{i \geq 1} = \{\Lambda_k\}_{k \geq 1},$$

and these eigenvalues are simple $\iff \nu = \sqrt{d_1/d_2} \notin \mathbb{Q}$.

Question: Is it possible to construct a **biorthogonal family** to $\{e^{-\Lambda_k t}\}_{k \geq 1}$ (in $L^2(0, \infty)$) which satisfies appropriate **bounds**??

8. The Kalman condition for a class of parabolic systems. Boundary controls

- ① We can apply Lemma 16 ($\eta \equiv 1$): there exists $\{q_k\}_{k \geq 1} \subset L^2(0, \infty)$ biorthogonal to $\{e^{-\Lambda_k t}\}_{k \geq 1}$ such that, for $C > 0$, one has

$$\|q_k\|_{L^2} \leq C|1 + \Lambda_k|^3 \prod_{\ell \geq 1, \ell \neq k} \left| \frac{1 + \Lambda_k / \Lambda_\ell^*}{1 - \Lambda_k / \Lambda_\ell} \right| := C|1 + \Lambda_k|^3 \mathcal{P}_k.$$

- ② The separability condition $|\Lambda_k - \Lambda_\ell| \geq \rho|k - \ell|$ ($\rho > 0$ a constant) does not hold. We cannot apply Proposition 17 and, in general, the following property fails:

“For any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ s.t. $\mathcal{P}_k \leq C(\varepsilon)e^{\varepsilon\Lambda_k}$, $\forall k \geq 1$ ”.

This property is crucial for proving the null controllability result for System (18) with boundary controls in $L^2(0, T)$ for **arbitrary final times** $T > 0$ (see the **scalar case** in 22).

Conclusion: This approach does not work when T is small.

8. The Kalman condition for a class of parabolic systems. Boundary controls

In a forthcoming paper

- **AMMAR-KHODJA, BENABDALLAH, G.-B., DE TERESA**, *Condensation index and necessary and sufficient conditions for the null controllability of abstract systems. Application to the boundary null controllability of coupled parabolic systems*, in preparation,

the authors prove for System (18) the following result:

Theorem

- ① Given $\nu = \sqrt{d_1/d_2} \notin \mathbb{Q}$, there exists $T_0 = T_0(\nu) \in [0, \infty]$ such that System (18) is **null controllable** at time T with controls $v \in L^2(0, T)$ $\iff T > T_0$.
- ② Given $T_0 \in [0, \infty]$, there exists a positive $\nu \notin \mathbb{Q}$ such that System (18) (for $d_1 = \nu^2$ and $d_2 = 1$) is **null controllable** at time T with controls $v \in L^2(0, T) \iff T > T_0$. ■

8. The Kalman condition for a class of parabolic systems. Boundary controls

We had two important differences between the controllability problem for **scalar** and **non-scalar** parabolic problems:

- 1 First difference, see slide 65, Section 4.??.
- 2 Second difference, see slide 68, Section 4.??.

Third difference with scalar problems

In general, we can get a null controllability result at time $T > 0$ for a **non-scalar parabolic** problem if T is **large enough**. ■

11. Further results

11. Further results: I. First and second order coupling terms

All previous results concern non-scalar parabolic equations with **zero order** coupling terms (a matrix A). For null controllability results for some 2×2 parabolic systems with first and second order coupling terms see [GUERRERO] SIAM J. Control Optim (2007). For system

$$(30) \quad \begin{cases} \partial_t y - \Delta y + cy + D \cdot \nabla y = \partial_{x_1}(w\theta_1) + v1_w & \text{in } Q_T, \\ \partial_t w - \Delta w + hw + K \cdot \nabla w = \Delta(y\theta_2) & \text{in } Q_T, \\ y = w = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0, \quad w(\cdot, 0) = w_0 & \text{in } \Omega, \end{cases}$$

with $c, h \in \mathbb{R}$ and $D, K \in \mathbb{R}^N$ and $\theta_1, \theta_2 \in C^2(\overline{\Omega})$, one has:

Theorem

Assume that there exists a nonempty open subset $\omega_2 \subset \omega$ and $C > 0$ such that $|\theta_2| \geq C > 0$ in ω_2 . Then System (30) is null controllable at any time $T > 0$.

Remark

Again the control open set ω have to meet the support of the function $|\theta_2|$.

11. Further results: II. Coupling matrices depending of x and t

In [BENABDALLAH,CRISTOFOL,GAITAN,DE TERESA] CRAS (2010), the following 3×3 control problem has been studied:

$$(31) \quad \begin{cases} \partial_t y = (\mathcal{L} + A)y + Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{L} = \text{diag}(L_1, L_2, L_2)$ with $(L_i)_{i=1,2}$ operators as in (2) satisfying (3) and (4), $A = (a_{ij})_{1 \leq i, j \leq 3} \in C^4(\overline{Q}_T; \mathcal{L}(\mathbb{R}^3))$, $B = (1, 0, 0)^* \in \mathbb{R}^3$, $v \in L^2(Q_T)$ is the control, and $y_0 \in L^2(\Omega; \mathbb{R}^3)$ is the initial condition.

11. Further results: II. Coupling matrices depending of x and t

Theorem

Suppose that a_{21} and a_{31} are time independent and $j \in \{2, 3\}$ such that $|a_{j1}(\cdot)| \geq C > 0$ in ω , $C > 0$. For $j \in \{2, 3\}$, we set $k_j = \frac{6}{j}$ and

$$B_{k_j} \in C^3(\overline{Q_T}; \mathbb{R}^N); \quad B_{k_j}^i := \sum_{\ell=1}^N \alpha_{i\ell}^{(2)} \left(\partial_l a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \partial_l a_{j1} \right), \quad 1 \leq i \leq N.$$

Assume that $\partial\omega \cap \partial\Omega = \gamma$, with $|\gamma| \neq 0$, and $B_{k_j} \cdot \nu \neq 0$ on γ , where ν is the outward unit normal vector. Then, System (31) is null controllable at time T .

Remark

A different **sufficient** condition has been obtained by K. Mauffrey. ■

11. Further results: III. Control domain and coupling terms

Consider the following system

$$(32) \quad \begin{cases} \partial_t y_1 = \Delta y_1 + \delta p y_2 + v 1_\omega & \text{in } Q_T, \\ \partial_t y_2 = p y_1 + \Delta y_2 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_{0,1}, \quad y_2(\cdot, 0) = y_{0,2} & \text{in } \Omega, \end{cases}$$

with p a smooth function and $\delta > 0$. In [ALABAU-BOUSSOIRA, LÉAUTAUD], CRAS (2011). One has:

Theorem

Let $p \geq 0$ on Ω . Assume that $\exists p_0 > 0$ and $\omega_p \subset \Omega$ satisfying the *Geometric Control Condition* (GCC) with $p \geq p_0$ in ω_p . Assume that ω also satisfies GCC. Then there exists $\delta_0 > 0$ such that for all $0 < \sqrt{\delta} \|p\|_{L^\infty(\Omega)} \leq \delta_0$ System (32) is null controllable at any positive time T .

11. Further results: III. Control domain and coupling terms

With the same kind of arguments, in the same paper, a new boundary control result is proved. For simplicity, consider

$$(33) \quad \begin{cases} \partial_t y_1 = \Delta y_1 + \delta p y_2 & \text{in } Q_T, \\ \partial_t y_2 = p y_1 + \Delta y_2 & \text{in } Q_T, \\ y_1 = b v, \quad y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_{0,1}, \quad y_2(\cdot, 0) = y_{0,2} & \text{in } \Omega, \end{cases}$$

where b is a function on $\partial\Omega$, $p \in L^\infty(\Omega)$ and $\delta > 0$. One has

Theorem

Let p satisfy assumptions of Theorem 11.1. Suppose that there $\exists \Gamma_b \subset \partial\Omega$ satisfying GCC and $b \geq b_0 > 0$ on Γ_b . Then there exists $\delta_0 > 0$ such that for all $0 < \sqrt{\delta} \|p\|_{L^\infty(\Omega)} \leq \delta_0$ System (33) is null controllable at any time $T > 0$.

11. Further results: III. Control domain and coupling terms

Even if the geometrical assumptions are too strong, these two theorems give the first examples on controllability of cascade system with coupling terms vanishing on the control domain. Moreover it also gives the first result on boundary control of two coupled parabolic equations for $N > 1$.

12. Comments and open problems

Most of the controllability results for parabolic systems are open.

A.- Let us consider the distributed controllability problem

$$(22) \quad \begin{cases} \partial_t y - D\Delta y = Ay + I_d v 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma, \quad y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega. \end{cases}$$

with $A \in \mathcal{L}(\mathbb{R}^n)$ (as before), $B = I_d$ and with $D \in \mathcal{L}(\mathbb{R}^n)$ a **non-symmetric** matrix such that the Jordan canonical form J is real and positive definite, i.e., $J \in \mathcal{L}(\mathbb{R}^n)$ and

$$\xi J \xi^* > 0, \quad \forall \xi \in \mathbb{R}^n, \xi \neq 0.$$

Some partial results by

E. FERNÁNDEZ-CARA, M. G.-B., L. DE TERESA, in preparation.

12. Comments and open problems

B.- The null controllability property of non-scalar parabolic problems with coupling matrices depending on x is open. A first problem:

Consider the distributed null controllability problem

$$\begin{cases} y_t - L_0 y = q(x) A_0 y + B v 1_\omega & \text{in } Q_T = \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma_T = \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0, & \text{in } \Omega, \end{cases}$$

with $A_0 \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$, $y_0 \in L^2(\Omega; \mathbb{R}^n)$, $v \in L^2(Q_T; \mathbb{R}^m)$ and q is a given scalar function.

Simple case:

A_0 is a “cascade matrix” and $q(x) = 1_{\mathcal{O}}$ with $\mathcal{O} \subset \Omega$ a new open set s.t.

$$\mathcal{O} \cap \omega = \emptyset.$$

12. Comments and open problems

C.- Kalman condition: Only in the cases presented here.

Other situations ?

D.- Boundary controllability for $N > 1$.

Reference:

[AMMAR-KHODJA, BENABDALLAH, G.-B., DE TERESA], *Recent results on the controllability of linear coupled parabolic problems: a survey*, *Mathematical Control and Related Fields* **1** (2011), no. 3, 267–306.

Very important: See the references therein.

Thanks for your attention !

¡ Gracias por vuestra atención !